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Generic modules of tame algebras over real closed fields

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This paper is dedicated to the memory of Professor Michael Butler

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Given a generically tame finite-dimensional algebra Λ over a real closed field, we describe the relations between the infinite families of indecomposable Λ-modules with bounded dimension and the generic Λ-modules. These are similar to those occurring for the algebraically closed field case, but the parametrizations are obtained over five particular centrally bounded principal ideal domains, instead of over rational algebras.

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1. Introduction

Denote by k a fixed ground field and let Λ be any k-algebra (associative, with unit element 1). Given a Λ-module G, recall that, by definition, the *endolength* of G is its length as a right $\mathrm{End}_\Lambda(G)^{op}$-module. The module G is called *generic* iff it is indecomposable, of infinite length as a Λ-module, but with finite endolength. The algebra Λ is called *generically tame* if, for each $d \in \mathbb{N}$, there is only a finite number of isomorphism classes of generic Λ-modules of endolength d. This notion was introduced by Crawley-Boevey in [9], providing a new definition of tameness, which coincides with the usual notion of tameness for finite-dimensional algebras over algebraically closed fields, but which makes sense for arbitrary algebras.

For example, consider the Kronecker k-algebra Λ, that is the path k-algebra of the quiver $\cdot \rightrightarrows \cdot$ Then, the Λ-module G determined by the representation

$$k(x) \underset{1}{\overset{x}{\rightrightarrows}} k(x)$$

is a generic Λ-module with $\mathrm{End}_\Lambda(G) \cong k(x)$. Moreover, if L is a finite field extension of k, then G^L is a generic Λ^L-module of the finite-dimensional k-algebra Λ^L, with $\mathrm{End}_{\Lambda^L}(G^L) \cong L(x)$, see [16], (5.1).

In this article we continue the study of the notion of generic tameness for finite-dimensional algebras Λ over perfect fields started in [4]. Our main results, which are obtained only for real closed fields, reveal some structural properties of generic Λ-modules. For the sake of simplicity, in this introduction, we assume that the ground field is the field of real numbers $\mathbb{R}$; the formulation of our results for real closed ground fields is essentially the same. Denote by $\mathbb{C}$ the complex numbers field and by $\mathbb{H}$ the real quaternions. Consider the skew polynomial algebras $\mathbb{R}[x]$, $\mathbb{C}[x]$, $\mathbb{H}[x]$, and $\mathbb{C}[x,\tau]$, where τ denotes the complex conjugation; consider also the principal ideal domain $\mathbb{D} = \mathbb{R}[x,y]/\langle y^2 + x^2 + 1\rangle$. Then, we have their corresponding skew fields of fractions $\mathbb{R}(x)$, $\mathbb{C}(x)$, $\mathbb{H}(x)$, $\mathbb{C}(x,\tau)$, and $\mathbb{E} = \mathbb{R}(x)[y]/\langle y^2 + x^2 + 1\rangle$. We will prove the following result.

Theorem 1.1. *Let Λ be a generically tame finite-dimensional $\mathbb{R}$-algebra and G a generic Λ-module. Then,*

1. *There is an algebra $\Gamma \in \{\mathbb{R}[x], \mathbb{C}[x], \mathbb{H}[x], \mathbb{C}[x,\tau], \mathbb{D}\}$ and a Λ-Γ-bimodule Z, finitely generated by the right, such that $G \cong Z \otimes_\Gamma Q$, where Q is the skew field of fractions of Γ.*
2. *The $\mathbb{R}$-algebra $\mathrm{End}_\Lambda(G)/\operatorname{rad}\mathrm{End}_\Lambda(G)$ is isomorphic to one of the following five algebras $\mathbb{R}(x), \mathbb{C}(x), \mathbb{H}(x), \mathbb{C}(x,\tau), \mathbb{E}$.*

The last result and the next one follow from the corresponding formulation for ditalgebras (an acronym for differential tensor algebras), which is proved in Section 10.

Theorem 1.2. *Let Λ be a generically tame finite-dimensional $\mathbb{R}$-algebra and let d be a non-negative integer. Then, there is a finite sequence of algebras $\Gamma_1, \dots, \Gamma_m \in \{\mathbb{R}[x], \mathbb{C}[x], \mathbb{H}[x], \mathbb{C}[x,\tau], \mathbb{D}\}$, and Λ-Γ_i-bimodules $Z_1, \dots, Z_m$, which are finitely generated as right Γ_i-modules, satisfying the following:*

1. *The functor $Z_i \otimes_{\Gamma_i} - : \Gamma_i\text{-Mod} \longrightarrow \Lambda\text{-Mod}$ preserves indecomposability and isomorphism classes, for any $i \in [1, m]$.*
2. *For each generic Λ-module G with $\operatorname{endol}(G) \leq d$, there is a unique $i \in [1, m]$ with $G \cong Z_i \otimes_{\Gamma_i} Q_i$, where Q_i is the skew field of fractions of Γ_i.*
3. *For almost every indecomposable Λ-module M with $\dim_k M \leq d$, we have $M \cong Z_i \otimes_{\Gamma_i} N$, for some $i \in [1, m]$ and $N \in \Gamma_i\text{-mod}$.*
4. *If $\{N_u\}_{u \in U}$ and $\{M_u\}_{u \in U}$ are infinite families of pairwise non-isomorphic indecomposable modules in $\Gamma_i\text{-mod}$ and $\Gamma_j\text{-mod}$, respectively, such that $Z_i \otimes_{\Gamma_i} N_u \cong Z_j \otimes_{\Gamma_j} M_u$ for all $u \in U$, then $i = j$.*

Since the indecomposable modules of finite length over bounded principal ideal domains are well understood, this result clarifies the relation between families of indecomposable Λ-modules with bounded dimension and generic Λ-modules. We provide more precise statements in 11.4.

The finiteness, in Theorem 1.1, of the list of possible isoclasses of the algebras $\operatorname{End}_\Lambda(G)/\operatorname{rad}\operatorname{End}_\Lambda(G)$, for all generic Λ-modules G and all finite-dimensional algebras Λ, is characteristic of the real closed or algebraically closed case. Indeed, if k is a non-algebraically closed and non-real closed field, as a consequence of Artin–Schreier Theorem, there is a family $\{L_n\}_{n \in \mathbb{N}}$ of finite field extensions of k with pairwise different degrees $[L_n : k]$. Then, if Λ denotes the Kronecker k-algebra described before, each finite-dimensional k-algebra Λ^{L_n} admits the generic Λ^{L_n}-module G^{L_n} with $\operatorname{End}_{\Lambda^{L_n}}(G^{L_n})/\operatorname{rad}\operatorname{End}_{\Lambda^{L_n}}(G^{L_n}) \cong L_n(x)$.

We have some final comments regarding our last section. In [4] we studied parametrizations of indecomposable Λ-modules with bounded dimension over perfect fields, but the study of generic modules and their interaction with families of indecomposables with bounded dimension is not yet solved for general perfect fields. We considered in this paper the more manageable case of real closed fields, where we were able to do this: the main results for finite-dimensional algebras Λ are Theorems 11.2 and 11.4. Our arguments use the fact that the degree of the algebraic closure of k is finite. The collateral result on the finiteness of the family of principal ideal domains involved in this analysis is not surprising, since the real closed fields k admit very few finite-dimensional division k-algebras up to isomorphism.

2. Constructibility and pregeneric modules

As usual, given any k-ditalgebra $\mathcal{A}$, we denote by $\mathcal{A}$-Mod the category of $\mathcal{A}$-modules. The full subcategory of $\mathcal{A}$-Mod formed by the finite-dimensional $\mathcal{A}$-modules is denoted by $\mathcal{A}$-mod. Let us first recall from [4] some terminology.

Definition 2.1. Let $\mathcal{A}$ be a layered ditalgebra, with layer (R, W), see [6], §4. Given $M \in \mathcal{A}$-Mod, denote by $E_M := \mathrm{End}_{\mathcal{A}}(M)^{op}$ its endomorphism algebra. Then, M admits a structure of R-E_M-bimodule, where $m \cdot (f^0, f^1) = f^0(m)$, for $m \in M$ and $(f^0, f^1) \in E_M$. By definition, the *endolength* of M, denoted by $\mathrm{endol}(M)$, is the length of M as a right E_M-module.

A module $M \in \mathcal{A}$-Mod is called *pregeneric* iff M is indecomposable, with finite endolength, but with infinite dimension over the ground field k. The ditalgebra $\mathcal{A}$ is called *pregenerically tame* iff, for each natural number d, there are only finitely many isoclasses of pregeneric $\mathcal{A}$-modules with endolength d.

Definition 2.2. Let $\mathcal{A} = (T, \delta)$ be a triangular ditalgebra, with layer (R, W), over any field k. Then,

1. $\mathcal{A}$ is called *admissible* iff $R \cong D_1 \times \cdots \times D_n$, for some finite-dimensional division k-algebras $D_1, \ldots, D_n$ and the R-R-bimodule W is finitely generated.
2. $\mathcal{A}$ is called *almost admissible* iff $R \cong M_{m_1}(D_1) \times \cdots \times M_{m_n}(D_n)$, for some finite-dimensional division k-algebras $D_1, \ldots, D_n$ and the R-R-bimodule W is finitely generated.

Definition 2.3. We say that an almost admissible ditalgebra $\mathcal{A}$, over a perfect field k, is *constructible* iff there is a finite sequence of reductions

$$\mathcal{D}^{\Lambda} = \mathcal{D} \longmapsto \mathcal{D}^{z_1} \longmapsto \mathcal{D}^{z_1 z_2} \longmapsto \cdots \longmapsto \mathcal{D}^{z_1 \cdots z_t},$$

where $\mathcal{D}^{\Lambda}$ is Drozd's ditalgebra of some finite-dimensional k-algebra Λ, as in [6], (19.1), and there is an isomorphism of layered ditalgebras $\mathcal{D}^{z_1 \cdots z_t} \cong \mathcal{A}$, for some finite set of reductions $\mathcal{D}^{z_1 \cdots z_{i-1}} \longmapsto \mathcal{D}^{z_1 \cdots z_i}$ of either of the types described in [4], (2.5) or in [4], (2.6) or in [4], (2.7). In this case, we say that $\mathcal{A}$ is *constructible from* Λ.

Remark 2.4. If the field k is perfect, any finite-dimensional k-algebra Λ splits over its radical. Then, Drozd's ditalgebra $\mathcal{D}$ of Λ is an almost admissible ditalgebra over k. It is admissible if and only if Λ is basic.

If K is any field extension of the perfect field k and $\mathcal{A}$ is an almost admissible ditalgebra over k, then the extended ditalgebra $\mathcal{A}^K$ is an almost admissible ditalgebra over K (see [6], §20). If the admissible ditalgebra $\mathcal{A}$ is constructible from a finite-dimensional k-algebra Λ, then $\mathcal{A}^K$ is an almost admissible ditalgebra constructible from the finite-dimensional K-algebra Λ^K, see [4], (4.3).

Notation 2.5. Given a finite-dimensional algebra Λ over any field k, denote by $\mathcal{P}(\Lambda)$ the category of morphisms between projective Λ-modules. Let J be the radical of Λ. Then $\mathcal{P}^1(\Lambda)$ denotes the full subcategory of $\mathcal{P}(\Lambda)$ whose objects are the morphisms $\alpha : P \longrightarrow Q$ with image contained in JQ, and $\mathcal{P}^2(\Lambda)$ denotes the full subcategory of $\mathcal{P}^1(\Lambda)$ whose objects are the morphisms $\alpha : P \longrightarrow Q$ with kernel contained in JP. If Λ splits over its radical, we can consider Drozd's ditalgebra $\mathcal{D} = \mathcal{D}^\Lambda$ and the usual equivalence functor $\Xi_\Lambda : \mathcal{D}\text{-Mod} \longrightarrow \mathcal{P}^1(\Lambda)$, see [6], (19.8).

Some of the following statements which include a constructibility assumption (remarkably 2.6, 5.2, and 6.1) are proved by transporting the corresponding known statement for modules over finite-dimensional algebras. We do not know if they can be proved directly in a more general context.

Theorem 2.6. *Assume that $\mathcal{A}$ is an almost admissible ditalgebra, which is constructible from a finite-dimensional algebra Λ, over a perfect field k. Then, for any pregeneric $\mathcal{A}$-module G, the algebra $\mathrm{End}_{\mathcal{A}}(G)$ is local and has nilpotent radical.*

Proof. From [10], (4.2) and [10], (4.4), the generic Λ-modules have local endomorphism algebras with nilpotent radical. Consider Drozd's ditalgebra $\mathcal{D} = \mathcal{D}^\Lambda$ and the composition of functors

$$\mathcal{D}\text{-Mod} \xrightarrow{\Xi_\Lambda} \mathcal{P}^1(\Lambda) \xrightarrow{\mathrm{Cok}} \Lambda\text{-Mod},$$

where Cok denotes the cokernel functor. Fix a pregeneric $\mathcal{D}$-module G. If $\Lambda = S \oplus J$ is a splitting over the radical J of Λ, then S is a finite-dimensional semisimple algebra. Then, since G has infinite k-dimension, the indecomposable object $\Xi_\Lambda(G)$ cannot be of the form $P \longrightarrow 0$. Hence, $\Xi_\Lambda(G) \in \mathcal{P}^2(\Lambda)$. It follows that $M = \mathrm{Cok}\,\Xi_\Lambda(G)$ is a generic Λ-module, see [4], (4.4). The functor $\mathrm{Cok}\,\Xi_\Lambda$ induces a morphism of algebras $\phi : \mathrm{End}_{\mathcal{D}}(G) \longrightarrow \mathrm{End}_\Lambda(M)$, which induces an isomorphism $\mathrm{End}_{\mathcal{D}}(G)/\operatorname{rad}\mathrm{End}_{\mathcal{D}}(G) \cong \mathrm{End}_\Lambda(M)/\operatorname{rad}\mathrm{End}_\Lambda(M)$, by [6], (31.6) and [6], (18.10). Then $\mathrm{End}_{\mathcal{D}}(G)$ is a local algebra. Using [6], (18.10)(2), it also follows that $\operatorname{rad}\mathrm{End}_{\mathcal{D}}(G)$ is nilpotent.

Now, adopt the notation of 2.3. Consider the isomorphism of layered ditalgebras $\xi : \mathcal{D}^{z_1\cdots z_t} \longrightarrow \mathcal{A}$ and the corresponding restriction functor $F_\xi : \mathcal{A}\text{-Mod} \longrightarrow \mathcal{D}^{z_1\cdots z_t}\text{-Mod}$. For $i \in [1,t]$, consider the corresponding reduction functor $F_i : \mathcal{D}^{z_1\cdots z_i}\text{-Mod} \longrightarrow \mathcal{D}^{z_1\cdots z_{i-1}}\text{-Mod}$. Then, the composition

$$F := F_1F_2\cdots F_tF_\xi : \mathcal{A}\text{-Mod} \longrightarrow \mathcal{D}\text{-Mod}$$

is a full and faithful functor which preserves pregeneric modules. Indeed, this is the case for each one of the factors by [4], (2.2) and [4], (2.5)–(2.7). For the factors of type F^X, we keep in mind that $S = \mathrm{End}(X)^{op}/\operatorname{rad}\mathrm{End}(X)^{op}$ is a finite-dimensional semisimple k-algebra and [6], (13.3), hence $N \in \mathcal{A}^X\text{-Mod}$ is an infinite-dimensional indecomposable if and only if $F^X(N) = X \otimes_S N$ is so. Then, given a pregeneric $\mathcal{A}$-module G, the module

$F(G)$ is also pregeneric and, since the functor F induces an isomorphism $\mathrm{End}_{\mathcal{A}}(G) \cong \mathrm{End}_{\mathcal{D}}(F(G))$, we get what we wanted. $\square$

3. Scalar restriction and pregeneric modules

Throughout this work, given a ditalgebra $\mathcal{A} = (T, \delta)$, we denote with a roman A the subalgebra $[T]_0$ of degree zero elements of the underlying graded algebra T of $\mathcal{A}$, see [6], §1. Then, the categories A-Mod and $\mathcal{A}$-Mod share the same class of objects, but there are more morphisms in $\mathcal{A}$-Mod. There is a canonical embedding functor $L_{\mathcal{A}} : A\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$, which is the identity on objects and maps each $f^0 \in \mathrm{Hom}_A(M, N)$ onto $L_{\mathcal{A}}(f^0) = (f^0, 0)$.

Lemma 3.1. *Let K be an extension of a field k. Assume that $\mathcal{A} = (T, \delta)$ is a k-ditalgebra, consider the K-ditalgebra $\mathcal{A}^K = (T^K, \delta^K)$ and the corresponding scalar extension functor $(-)^K : \mathcal{A}\text{-Mod} \longrightarrow \mathcal{A}^K\text{-Mod}$, as in [6], (20.2). We have a morphism of k-ditalgebras $\xi : \mathcal{A} \longrightarrow \mathcal{A}^K$, given by $\xi(t) = t \otimes 1$, for $t \in T$. Then, we have the associated restriction functor*

$$F_\xi : \mathcal{A}^K\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod},$$

which we call the scalar restriction functor. *It is a faithful functor satisfying the following:*

1. *The functor F_ξ is a right adjoint to $(-)^K$. The corresponding natural isomorphism is given for $M \in \mathcal{A}\text{-Mod}$ and $N \in \mathcal{A}^K\text{-Mod}$ by*

$$\eta : \mathrm{Hom}_{\mathcal{A}^K}\big(M^K, N\big) \longrightarrow \mathrm{Hom}_{\mathcal{A}}\big(M, F_\xi(N)\big),$$

   with $\eta(f) = (\eta(f)^0, \eta(f)^1)$, where $\eta(f)^0[m] = f^0[m \otimes 1]$ and $\eta(f)^1(v)[m] = f^1(v \otimes 1)[m \otimes 1]$, for $v \in V := [T]_1$ and $m \in M$.
2. *For $M \in \mathcal{A}\text{-Mod}$, we have $F_\xi(M^K) \cong \coprod_{\mathbb{B}} M$, where $\mathbb{B}$ is a k-basis for K.*
3. *For $N \in \mathcal{A}^K\text{-Mod}$, we have $[F_\xi(N)]^K \cong N \otimes_K K \otimes_k K$.*

Proof. Take a morphism $f \in \mathrm{Hom}_{\mathcal{A}^K}(M, N)$. If $F_\xi(f^0, f^1) = (f^0, f^1\xi_1) = 0$, then $f^0 = 0$ and $f^1(v \otimes 1) = f^1\xi(v) = 0$, for any element $v \in V$. But $f^1 \in \mathrm{Hom}_{A^K\text{-}A^K}(V^K, \mathrm{Hom}_K(M, N))$, thus $f^1(v \otimes \lambda) = f^1(v \otimes 1)\lambda = 0$, and also $f^1 = 0$. Thus, $f = 0$ and F_ξ is a faithful functor.

(1): Assume $M \in \mathcal{A}$-Mod and $N \in \mathcal{A}^K$-Mod. Then, the inverse of η is given by the linear map

$$\psi : \mathrm{Hom}_{\mathcal{A}}\big(M, F_\xi(N)\big) \longrightarrow \mathrm{Hom}_{\mathcal{A}^K}\big(M^K, N\big),$$

with $\psi(g) = (\psi(g)^0, \psi(g)^1)$, where $\psi(g)^0[m \otimes \lambda] = \lambda g^0[m]$ and $\psi(g)^1(v \otimes \lambda)[m \otimes \lambda'] = \lambda\lambda' g^1(v)[m]$, for $v \in V$, $m \in M$, and $\lambda, \lambda' \in K$. The verification of this, together with the fact that η and ψ are indeed well defined linear maps (natural in M and N) is straightforward.

(2): This is clear.

(3): An A^K-module N can be considered naturally as an A-K-bimodule (with underlying A-module $F_\xi(N)$). Thus, $F_\xi(N) \cong N \otimes_K K$ in A-Mod. Therefore, $[F_\xi(N)]^K \cong N \otimes_K K \otimes_k K$ in A^K-Mod. □

The following three results adapt Lemmas 3.2(b) and 3.3(b) of [16] to our ditalgebras context.

Lemma 3.2. *Let $\mathcal{A}$ be an almost admissible ditalgebra over any field k and take any field extension K of k. Given $M, N \in \mathcal{A}$-Mod, consider the natural map $\alpha : \mathrm{Hom}_{\mathcal{A}}(M,N)^K \longrightarrow \mathrm{Hom}_{\mathcal{A}^K}(M^K,N^K)$, as in [4], (5.1). Then, for every finitely generated R^K-submodule Z of M^K and every $f \in \mathrm{Hom}_{\mathcal{A}^K}(M^K,N^K)$ there exists $g \in \mathrm{Hom}_{\mathcal{A}}(M,N)^K$ such that f^0 and $\alpha(g)^0$ coincide on Z.*

Proof. Let $\mathbb{B}$ be a k-basis of K. Consider the isomorphism $\zeta : F_\xi(N^K) \longrightarrow \coprod_{\mathbb{B}} N$, and the adjunction isomorphism $\eta : \mathrm{Hom}_{\mathcal{A}^K}(M^K,N^K) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(M, F_\xi(N^K))$, as in (3.1). For $b \in \mathbb{B}$, denote by $\pi_b : \coprod_{\mathbb{B}} N \longrightarrow N$ the canonical projection on the copy b of N, and by $\sigma_b : N \longrightarrow \coprod_{\mathbb{B}} N$ the corresponding canonical injection. Given $f \in \mathrm{Hom}_{\mathcal{A}^K}(M^K,N^K)$, we will consider $g := \sum_{b\in\mathbb{B}_0} \pi_b\zeta\eta(f) \otimes b \in \mathrm{Hom}_{\mathcal{A}}(M,N)^K$, where $\mathbb{B}_0$ is a finite subset of $\mathbb{B}$ chosen as follows. From Z we can obtain a finitely generated R-submodule Z_0 of M such that $Z \subseteq Z_0^K$. Choose a k-basis $z_1, \dots, z_s$ for Z_0, then there is a finite subset $\mathbb{B}_0$ of $\mathbb{B}$ such that $(\zeta\eta(f))^0[z_i] \in \coprod_{\mathbb{B}_0} N$, for all $i \in [1,s]$. Thus, we can write $(\zeta\eta(f))^0[z_i] = \sum_{b\in\mathbb{B}_0} \sigma_b^0(n_{i,b})$, where $n_{i,b} \in N$. Equivalently, $f^0(z_i \otimes 1) = \sum_{b\in\mathbb{B}_0} n_{i,b} \otimes b$.

Now we show that $\alpha(g)^0$ and f^0 coincide on Z_0^K, hence on Z. Take a typical generator $z_i \otimes c$ of the K-vector space Z_0^K. Then,

$$\begin{aligned} \alpha(g)^0[z_i \otimes c] &= \sum_{b\in\mathbb{B}_0} \big(\pi_b\zeta\eta(f)\big)^0[z_i] \otimes bc \\ &= \sum_{b\in\mathbb{B}_0} \pi_b^0\Bigg(\sum_{b'\in\mathbb{B}_0} \sigma_{b'}^0(n_{i,b'})\Bigg) \otimes bc \\ &= \sum_{b\in\mathbb{B}_0} n_{i,b} \otimes bc = f^0(z_i \otimes 1)c = f^0(z_i \otimes c). \qquad \square \end{aligned}$$

Lemma 3.3. *Let $\mathcal{A}$ be an almost admissible ditalgebra over any field k, and take any field extension K of k. Assume that $M, N \in \mathcal{A}$-Mod satisfy that M^K and N^K have a common non-zero direct summand. If $\mathrm{End}_{\mathcal{A}}(M)$ is local with nilpotent radical, the module M is a direct summand of N in $\mathcal{A}$-Mod.*

Proof. We follow Kasjan's argument in [16], (3.3). If M^K and N^K have a common direct summand then there exist morphisms $f \in \mathrm{Hom}_{\mathcal{A}^K}(M^K,N^K)$ and $g \in$

$\mathrm{Hom}_{\mathcal{A}^K}(N^K, M^K)$ such that gf is a non-zero idempotent of M^K. Choose $y \in M^K$ such that $(gf)^0(y) = y \neq 0$. Apply 3.2 to obtain $f_1, \dots, f_a \in \mathrm{Hom}_{\mathcal{A}}(M, N)$; $g_1, \dots, g_b \in \mathrm{Hom}_{\mathcal{A}}(N, M)$; and scalars $\lambda_1, \dots, \lambda_a, \mu_1, \dots, \mu_b \in K$, with $f^0(y) = (\alpha[\sum_{i=1}^a f_i \otimes \lambda_i])^0(y)$ and $g^0(f^0(y)) = (\alpha[\sum_{j=1}^b g_j \otimes \mu_j])^0(f^0(y))$. The morphism α satisfies $\alpha[h \otimes \lambda] = h^K \lambda$, for $h \in \mathrm{Hom}_{\mathcal{A}}(M, N)$ and $\lambda \in K$. Thus, it gives a morphism of K-algebras $\mathrm{End}_{\mathcal{A}}(M)^K \to \mathrm{End}_{\mathcal{A}^K}(M^K)$ and $0 \neq y = (gf)^0(y) = (\alpha[\sum_{j=1}^b \sum_{i=1}^a g_j f_i \otimes \mu_j \lambda_i])^0(y)$. It follows that $\sum_{j=1}^b \sum_{i=1}^a g_j f_i \otimes \mu_j \lambda_i$ is not nilpotent in $\mathrm{End}_{\mathcal{A}}(M)^K$. But, by assumption, the algebra $\mathrm{End}_{\mathcal{A}}(M)$ is local with nilpotent radical, hence, there exist i_0 and j_0 such that $g_{j_0} f_{i_0}$ is not nilpotent. Then, this composition $g_{j_0} f_{i_0}$ is an isomorphism. So, M is isomorphic to a direct summand of N, because idempotents split in $\mathcal{A}$-Mod. □

Corollary 3.4. *Let $\mathcal{A}$ be an almost admissible constructible ditalgebra over a perfect field k and take any field extension K of k. Assume that $M, N \in \mathcal{A}$-Mod satisfy that M^K and N^K have a common non-zero direct summand. If M is indecomposable with finite endolength, then M is a direct summand of N in $\mathcal{A}$-Mod.*

Proof. We have that $\mathrm{End}_{\mathcal{A}}(M)$ is local with nilpotent radical: If M has infinite dimension, it is pregeneric and we can apply 2.6; if M is finite-dimensional, it follows from [6], (5.12). □

Lemma 3.5. *Let $\mathcal{A}$ be a layered ditalgebra over a field k and let K be a finite field extension of k. Then, for $M \in \mathcal{A}$-Mod, we have that*

$$\mathrm{endol}(M) \leq \mathrm{endol}\big(M^K\big) \leq [K : k] \times \mathrm{endol}(M).$$

Proof. We show that Kasjan's argument in [16], (3.3) works also in this context. The natural map $\alpha : \mathrm{Hom}_{\mathcal{A}}(M, N)^K \longrightarrow \mathrm{Hom}_{\mathcal{A}^K}(M^K, N^K)$ is an isomorphism. Indeed, from [6], (4.10), we have a commutative diagram with exact columns

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathrm{Hom}_{\mathcal{A}}(M, N)^K & \xrightarrow{\alpha} & \mathrm{Hom}_{\mathcal{A}^K}(M^K, N^K) \\
{\scriptstyle \sigma_{\mathcal{A}} \otimes 1} \downarrow & & \downarrow {\scriptstyle \sigma_{\mathcal{A}^K}} \\
\mathrm{Phom}_{R\text{-}W}(M, N)^K & \xrightarrow{\alpha'} & \mathrm{Phom}_{R^K\text{-}W^K}(M^K, N^K) \\
{\scriptstyle \partial_{\mathcal{A}} \otimes 1} \downarrow & & \downarrow {\scriptstyle \partial_{\mathcal{A}^K}} \\
\mathrm{Hom}_R(W_0 \otimes_R M, N)^K & \xrightarrow{\alpha''} & \mathrm{Hom}_{R^K}(W_0^K \otimes_{R^K} M^K, N^K),
\end{array}$$

where α' and α'' are the corresponding natural morphisms. Since K is a finite extension of k, we can apply [16], (3.2)(a) to obtain that the maps α' and α'' are isomorphisms and, therefore, α is an isomorphism too.

It follows that, given any $\mathrm{End}_{\mathcal{A}}(M)$-submodule N of M, we have that N^K is an $\mathrm{End}_{\mathcal{A}^K}(M^K)$-submodule of M^K. The first inequality follows from the fact that the map $N \mapsto N^K$ sends proper chains of $\mathrm{End}_{\mathcal{A}}(M)$-submodules of M onto proper chains of $\mathrm{End}_{\mathcal{A}^K}(M^K)$-submodules of M^K.

For the second inequality, make $E = \mathrm{End}_{\mathcal{A}}(M)^{op}$. Then, the module M^K is an E-module through the k-algebra morphism $\beta : E \longrightarrow \mathrm{End}_{\mathcal{A}^K}(M^K)^{op}$ induced by the scalar extension functor $(-)^K$. Then, we have $\mathrm{endol}(M^K) \leq \ell_E(M^K) = \ell_E(M \otimes_k K) = \ell_E(M^{[K:k]}) = [K : k] \times \ell_E(M) = [K : k] \times \mathrm{endol}(M)$. □

Lemma 3.6. *Let $\mathcal{A}$ be a layered k-ditalgebra and K any field extension of k. Then, for any $N \in \mathcal{A}^K$-Mod, we have that $\mathrm{endol}(F_\xi(N)) \leq \mathrm{endol}(N)$.*

Proof. It follows from [4], (2.2). □

Lemma 3.7. *If $\mathcal{A}$ is a k-ditalgebra, K is a finite separable field extension of k, and $N \in \mathcal{A}^K$-Mod, then N is a direct summand of $[F_\xi(N)]^K$.*

Proof. We proceed as in [16], (4.1). The K-K-bimodule $K \otimes_k K$ is semisimple and the trivial bimodule K is a direct summand of $K \otimes_k K$. Then, from 3.1(3), any $N \in \mathcal{A}^K$-Mod is a direct summand of $N \otimes_K K \otimes_k K \cong [F_\xi(N)]^K$. □

Lemma 3.8. *If $\mathcal{A}$ is an almost admissible constructible k-ditalgebra, K is a finite field extension of the perfect field k, and G is a pregeneric $\mathcal{A}$-module, then G^K admits no finite-dimensional direct summands.*

Proof. Suppose that $G^K \cong M \oplus N$, where N is indecomposable and finite-dimensional over K. Thus, $F_\xi(N)$ is finite-dimensional and admits a decomposition $F_\xi(N) \cong \bigoplus_{i=1}^n L_i$, with each $L_i \in \mathcal{A}$-mod indecomposable. Thus, $[F_\xi(N)]^K \cong \bigoplus_{i=1}^n {L_i}^K$. But, from 3.7, we know that N is an indecomposable direct summand of this module. Thus N is a direct summand of some L_i^K. From 3.4, we have that G is a direct summand of L_i, a contradiction. □

4. Endolength and realizations

In this section, which is a little technical, we recollect some basic properties of the reduction functors of type F^X, and we give a brief discussion of some elementary properties of realizations of pregeneric modules (see 4.8).

Reminder 4.1. We recall some terminology from [6] and [3]. Let $\mathcal{A} = (T, \delta)$ be any ditalgebra with layer (R, W). Assume we have R-R-bimodule decompositions $W_0 = W_0' \oplus W_0''$ and $W_1 = W_1' \oplus W_1''$. Consider the subalgebra T' of T generated by R and $W' = W_0' \oplus W_1'$, and the subalgebra A' of A generated by R and W_0'. Let us assume furthermore that $\delta(W_0') \subseteq A'W_1'A'$ and $\delta(W_1') \subseteq A'W_1'A'W_1'A'$. Then, the differential

δ on T restricts to a differential δ' on the algebra T' and we obtain a new ditalgebra $\mathcal{A}' = (T', \delta')$ with layer (R, W'). A layered ditalgebra $\mathcal{A}'$ is called a *proper subditalgebra of* $\mathcal{A}$ if it is obtained from an R-R-bimodule decomposition of W as we have just described.

A proper subditalgebra $\mathcal{A}'$ of a triangular ditalgebra $\mathcal{A}$ is called *initial* when its triangular filtrations coincide with the first terms of the triangular filtrations of $\mathcal{A}$, see [6], (14.8).

The inclusion $r : T' \longrightarrow T$ yields a morphism of ditalgebras $r : \mathcal{A}' \longrightarrow \mathcal{A}$ and, hence, a *restriction functor* (see [6], (2.4))

$$R^{\mathcal{A}}_{\mathcal{A}'} := F_r : \mathcal{A}\text{-}\mathrm{Mod} \longrightarrow \mathcal{A}'\text{-}\mathrm{Mod}.$$

The projection $\pi : A = [T]_0 \longrightarrow [T']_0 = A'$ yields an *extension functor*

$$E^{A}_{A'} := F_\pi : A'\text{-}\mathrm{Mod} \longrightarrow A\text{-}\mathrm{Mod}.$$

When there is no danger of confusion, we forget subindices and superindices in restriction and extension functors.

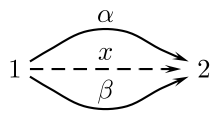
Let $\mathcal{A} = (T, \delta)$ be a ditalgebra with layer (R, W). Then, an algebra B is called a *proper subalgebra of* $\mathcal{A}$ if and only if $B = [T']_0$, for some proper subditalgebra $\mathcal{A}' = (T', \delta')$ of $\mathcal{A}$ associated to R-R-bimodule decompositions $W_0 = W'_0 \oplus W''_0$ and $W_1 = W'_1 \oplus W''_1$, where $W'_1 = 0$. In this case, the ditalgebra $\mathcal{B} := \mathcal{A}'$ is essentially the same as the algebra B, we also call $\mathcal{B}$ a proper subalgebra of $\mathcal{A}$, the module categories B-Mod and $\mathcal{B}$-Mod are canonically identified through the canonical embedding functor $L_{\mathcal{B}} : B\text{-}\mathrm{Mod} \longrightarrow \mathcal{B}\text{-}\mathrm{Mod}$.

Remark 4.2. With the notation of 4.1, we notice that, sometimes, it is possible to define an extension functor $E^{\mathcal{A}}_{\mathcal{A}'} : \mathcal{A}'\text{-}\mathrm{Mod} \longrightarrow \mathcal{A}\text{-}\mathrm{Mod}$ such that the following square commutes

$$\begin{array}{ccc} A\text{-}\mathrm{Mod} & \xrightarrow{L_A} & \mathcal{A}\text{-}\mathrm{Mod} \\ {\scriptstyle E^{A}_{A'}}\uparrow & & \uparrow{\scriptstyle E^{\mathcal{A}}_{\mathcal{A}'}} \\ A'\text{-}\mathrm{Mod} & \xrightarrow{L_{\mathcal{A}'}} & \mathcal{A}'\text{-}\mathrm{Mod}, \end{array}$$

which is of course a very useful property. This has been done and exploited in [5] for convex subditalgebras $\mathcal{A}'$ of seminested ditalgebras $\mathcal{A}$. In [4], this was done and exploited for proper subalgebras $\mathcal{A}'$ of layered ditalgebras $\mathcal{A}$. When $\mathcal{A}'$ is a minimal subditalgebra of a seminested ditalgebra $\mathcal{A}$, we can consider the functor $E^{\mathcal{A}}_{\mathcal{A}'}$ mapping each $\mathcal{A}'$-morphism (f^0, f^1) onto $(f^0, 0)$. This is crucial in the proof of our Theorem 5.4.

Unfortunately, this is not always possible. We give a simple example to illustrate this fact, where we provide $\mathcal{A}'$-modules M and N with $M \cong N$ in $\mathcal{A}'$-Mod but $E^{\mathcal{A}}_{\mathcal{A}'}(M) \ncong E^{\mathcal{A}}_{\mathcal{A}'}(N)$ in $\mathcal{A}$-Mod. Consider the nested k-ditalgebra $\mathcal{A} = (T_R(W), \delta)$ defined by the following bigraph



with $\delta(\alpha) = x = \delta(\beta)$ and $\delta(x) = 0$. Thus, $\mathcal{A}$ has layer (R, W) with $R = k \times k$, $W_0 = k\alpha \oplus k\beta$, $W_1 = kx$ and $W = W_0 \oplus W_1$. It admits the triangular filtrations $0 \subseteq k\alpha \subseteq k\alpha \oplus k\beta = W_0$ and $0 \subseteq kx = W_1$. Consider the proper subditalgebra $\mathcal{A}'$ of $\mathcal{A}$ defined by the R-R-bimodules $W_0' = k\alpha$, $W_0'' = k\beta$, and $W_1' = kx$. Then, $\mathcal{A}'$ is an initial subditalgebra of $\mathcal{A}$. The A'-modules M and N given by the representations $k \xrightarrow{1} k$ and $(k \longrightarrow 0) \oplus (0 \longrightarrow k)$ give the wanted example.

Definition 4.3. Let $\mathcal{A}$ be an almost admissible ditalgebra, with layer (R, W). Suppose E is some k-algebra and consider the decomposition $1 = \sum_{i=1}^n e_i$ of the unit of R as a sum of central primitive orthogonal idempotents e_i of R. Given $M \in \mathcal{A}$-E-Mod, as in [6], (21.1), we can consider its *length vector*

$$\underline{\ell}_E(M) = \big(\ell_E(e_1 M), \ldots, \ell_E(e_n M)\big).$$

Lemma 4.4. *Let k be a perfect field and let $\mathcal{A}$ be an almost admissible ditalgebra with layer (R, W). Assume that $\mathcal{A}^X$ is obtained from $\mathcal{A}$ by reduction, using a $\mathcal{B}$-module X, where $\mathcal{B}$ is an initial subalgebra of $\mathcal{A}$ and X is a finite direct sum of pairwise non-isomorphic finite-dimensional indecomposable $\mathcal{B}$-modules, see [6], (12.9). Then, $\Gamma = \mathrm{End}_B(X)^{op}$ admits the splitting $\Gamma = S \oplus P$, where P is the radical of Γ, and $\mathcal{A}^X$ is an admissible ditalgebra with triangular layer (S, W^X). We shall denote by $\{e_i\}_{i=1}^n$ (respectively $\{f_j\}_{j=1}^m$) the orthogonal primitive central idempotents given by the unit decomposition of R (resp. of S). Then, for any k-algebra E, we have:*

1. *The associated functor $F_X : \mathcal{A}^X\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$, see [6], (12.10), is full and faithful. Moreover, $\mathrm{endol}(N) \leq \mathrm{endol}(F_X(N))$, for any $N \in \mathcal{A}^X\text{-Mod}$.*
2. *The induced functor $F_X^E : \mathcal{A}^X\text{-}E\text{-Mod} \longrightarrow \mathcal{A}\text{-}E\text{-Mod}$, see [6], (21.3), satisfies that $\ell_E(N) \leq \ell_E(F_X^E(N))$, for any $N \in \mathcal{A}^X\text{-}E\text{-Mod}$.*
3. *For any $N \in \mathcal{A}^X\text{-}E\text{-Mod}$, we have $\underline{\ell}_E(F_X^E(N))^t = [X]\underline{\ell}_E(N)^t$, where $[X]$ is the matrix with non-negative integral entries $[X]_{i,j} = \dim_{Sf_j}(e_i X f_j)$.*

Proof. The finite-dimensional algebra Γ splits over its radical because k is a perfect field. The algebra S is basic because the indecomposable direct summands of X are pairwise non-isomorphic. From [6], (5.6), we know that $\mathcal{A}$ is a Roiter ditalgebra. Since $\mathcal{B}$ is a subalgebra of $\mathcal{A}$, the $\mathcal{B}$-module X is admissible, see [6], (12.4). The $\mathcal{B}$-module X is complete by [6], (13.3), and it is triangular by [6], (17.4). Then, $\mathcal{A}^X$ is a triangular ditalgebra, its natural triangular structure is described in [6], (14.10). From [6], (13.5), we know that F_X is full and faithful.

Recall that the triangular $\mathcal{B}$-module X admits a right additive B-S-bimodule filtration $\mathcal{F}(X):\ 0 = X^0 \subseteq X^1 \subseteq \cdots \subseteq X^{\ell_X} = X$, such that $X^t P \subseteq X^{t-1}$, for all $t \in [1, \ell_X]$. We know that any $N \in \mathcal{A}^X$-E-Mod is an S-E-bimodule via $ne := \alpha_N(e)^0(n)$, where $\alpha_N : E \longrightarrow \mathrm{End}_{\mathcal{A}^X}(N)^{op}$ comes from the given $\mathcal{A}$-E-bimodule structure of $N \in \mathcal{A}^X$-E-Mod, $n \in N$, and $e \in E$, see [6], (21.1). Thus, each $e_iX^t \otimes_S N$ inherits a natural structure of an Re_i-E-bimodule. Namely, $(x \otimes n) \star e := x \otimes (ne)$, for $x \in e_iX^t$ and $n \in N$. We denote the length of submodules or quotients of these modules with the symbol $\ell^\star_E$.

From [6], (12.10) and [6], (21.3), the morphism $\alpha_{F_X(N)} : E \longrightarrow \mathrm{End}_{\mathcal{A}}(F_X(N))^{op}$ satisfies, for $e \in E$, $n \in N$, and $x \in X$, the equality $(\alpha_{F_X(N)}(e))^0[x \otimes n] = F_X(\alpha_N(e))^0[x \otimes n] = x \otimes \alpha_N(e)^0(n) + \sum_\xi xp_\xi \otimes \alpha_N(e)^1(\gamma_\xi)[n]$, where $(p_\xi, \gamma_\xi)_\xi$ is a dual basis for the projective right S-module P. Consider the structure of Re_i-E-bimodule on $e_iF_X(N) = e_iX \otimes_S N$ determined by the $\mathcal{A}$-E-bimodule $F^E_X(N)$, that is $(x \otimes n) \cdot e = \alpha_{F_X(N)}(e)^0[x \otimes n]$, for $x \in e_iX$ and $n \in N$. From the previous formula for $\alpha_{F_X(N)}(e)^0$, we immediately obtain that each $e_iX^t \otimes_S N$ is an Re_i-E-subbimodule of $e_iF_X(N)$. We write the length of submodules or quotients of these modules with the symbol ℓ_E.

We can show that $\ell^\star_E(e_iX^t \otimes_S N) = \ell_E(e_iX^t \otimes_S N)$, for any $t \in [0, \ell_X]$, as in the proof of [6], (25.7). Then, for $t = \ell_X$, we have $X = X^t$, and

$$\begin{aligned}\ell_E(e_iX \otimes_S N) &= \ell^\star_E(e_iX \otimes_S N) = \ell^\star_E(\oplus_j e_iXf_j \otimes_S f_jN)\\ &= \sum_j \ell^\star_E(e_iXf_j \otimes_{Sf_j} f_jN)\\ &= \sum_j \dim_{Sf_j}[e_iXf_j]\ell_E(f_jN).\end{aligned}$$

We have an induced functor $F^E_X : \mathcal{A}^X$-E-Mod $\longrightarrow \mathcal{A}$-$E$-Mod, which is full and faithful by [6], (21.3). Each $f_j = (f^0_j, 0) \in \mathrm{End}_{\mathcal{B}}(X)^{op}$ is a non-zero idempotent, thus $Xf_j \neq 0$, and $e_iXf_j \neq 0$, for some i. Then, we have $\ell_E(F^E_X(N)) = \sum_i \ell_E(e_iX \otimes_S N) = \sum_{i,j} \dim_{Sf_j}[e_iXf_j]\ell_E(f_jN) \geq \sum_j \ell_E(f_jN) = \ell_E(N)$. We have proved (2) and (3).

Finally, given $N \in \mathcal{A}^X$-Mod, consider the algebra $E := \mathrm{End}_{\mathcal{A}^X}(N)^{op}$. Then, we have an isomorphism $E \cong \mathrm{End}_{\mathcal{A}}(F_X(N))^{op}$ induced by F_X, which provides, by restriction the structure of right E-module of $F^E_X(N)$. Then, $\mathrm{endol}(N) = \ell_E(N) \leq \ell_E(F^E_X(N)) = \mathrm{endol}(F_X(N))$, and (1) holds. □

The proof of the following statement is similar to that of the preceding one (see also [6], (25.7)).

Lemma 4.5. *Let $\mathcal{A}$ be a seminested ditalgebra with layer (R, W), over an algebraically closed field. Assume that $\mathcal{A}^X$ is a seminested ditalgebra obtained from $\mathcal{A}$ by reduction, using a complete triangular admissible $\mathcal{B}$-module X, where $\mathcal{B}$ is an initial subalgebra of $\mathcal{A}$. Thus, $\mathcal{A}^X$ has layer (S, W^X), where $\Gamma = \mathrm{End}_{\mathcal{B}}(X)^{op}$ admits the splitting $\Gamma = S \oplus P$. We shall denote by $\{e_i\}^n_{i=1}$ (respectively $\{f_j\}^m_{j=1}$) the orthogonal primitive central idempotents given by the unit decomposition of R (resp. of S). Then, for any k-algebra E, we have:*

1. *The associated functor* $F_X : \mathcal{A}^X\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$*, see [6], (12.10), is full and faithful. Moreover,* $\mathrm{endol}(N) \leq \mathrm{endol}(F_X(N))$*, for any* $N \in \mathcal{A}^X\text{-Mod}$*.*
2. *The induced functor* $F_X^E : \mathcal{A}^X\text{-}E\text{-Mod} \longrightarrow \mathcal{A}\text{-}E\text{-Mod}$*, see [6], (21.3), satisfies that* $\ell_E(N) \leq \ell_E(F_X^E(N))$*, for any* $N \in \mathcal{A}^X\text{-}E\text{-Mod}$*.*
3. *For any* $N \in \mathcal{A}^X\text{-}E\text{-Mod}$*, we have* $\underline{\ell}_E(F_X^E(N))^t = [X]\underline{\ell}_E(N)^t$*, where* $[X]$ *is the matrix with non-negative integral entries* $[X]_{i,j} = \mathrm{rank}_{Sf_j}(e_i X f_j)$*.*

Lemma 4.6. *Let* $\mathcal{A}$ *be a layered ditalgebra over a perfect field* k*. Assume that* $\mathcal{A}^X$ *is the layered ditalgebra obtained from* $\mathcal{A}$ *by reduction, using a finite-dimensional* $\mathcal{B}$*-module* X*, where* $\mathcal{B}$ *is a proper subalgebra of* $\mathcal{A}$*. Assume that* X *is a complete triangular admissible* $\mathcal{B}$*-module and consider the associated functor* $F_X : \mathcal{A}^X\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$*. Then, there is a constant* C *such that*

1. *For any* $N \in \mathcal{A}^X\text{-Mod}$*, we have* $\mathrm{endol}(N) \leq C \times \mathrm{endol}(F_X(N))$*.*
2. *For any* k*-algebra* E*, the induced functor* $F_X^E : \mathcal{A}^X\text{-}E\text{-Mod} \longrightarrow \mathcal{A}\text{-}E\text{-Mod}$ *satisfies that* $\ell_E(N) \leq C \times \ell_E(F_X^E(N))$*, for any* $N \in \mathcal{A}^X\text{-}E\text{-Mod}$*.*

Proof. The finite-dimensional algebra $\Gamma = \mathrm{End}_B(X)^{op}$ admits the splitting $\Gamma = S \oplus P$, where P is the radical of Γ, because k is perfect. Thus, S is a semisimple algebra. Consider the canonical central primitive orthogonal idempotents $\{f_j\}_{j=1}^m$ of S. Thus, each Sf_j is a simple algebra and we can choose a constant C such that $(Xf_j)^{(C)} \cong Sf_j \oplus Z_j$, for all j, for some appropiate right Sf_j-modules Z_j.

Given a k-algebra E and $N \in \mathcal{A}^X$-E-Mod, as in the last proof, we can show that, in order to calculate the length of the E-module $F_X^E(N)$, we can calculate the length of the E-module $X \otimes_S N$ with the usual action of E on the second tensor factor N. Then, we obtain the following: $\ell_E(N) = \ell_E(\bigoplus_j f_j N) = \sum_j \ell_E(f_j N) = \sum_j \ell_E(Sf_j \otimes_{Sf_j} f_j N) \leq \sum_j \ell_E((Xf_j)^{(C)} \otimes_{Sf_j} f_j N) = C \times \sum_j \ell_E(Xf_j \otimes_{Sf_j} f_j N) = C \times \ell_E(X \otimes_S N) = C \times \ell_E(F_X^E(N))$. Then, (2) holds. Item (1) is obtained from (2) as in the last proof. □

The following fact applies in particular to the situations considered in the three preceding lemmas.

Lemma 4.7. *Let* $\mathcal{A}$ *be a layered ditalgebra over a field* k*. Assume that* $\mathcal{A}^X$ *is the layered ditalgebra obtained from* $\mathcal{A}$ *by reduction, using a complete triangular admissible* $\mathcal{B}$*-module* X*, where* $\mathcal{B}$ *is a proper subalgebra of* $\mathcal{A}$*. Thus,* $\mathcal{A}^X$ *has layer* (S, W^X)*, where* $\Gamma = \mathrm{End}_{\mathcal{B}}(X)^{op}$ *admits the splitting* $\Gamma = S \oplus P$*. Denote by* $\mu(X)$ *the number of generators in a set of generators of the right* S*-module* X *with minimal cardinality. Then, we have:*

1. *The associated functor* $F_X : \mathcal{A}^X\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$ *is full and faithful. Moreover, for any* $N \in \mathcal{A}^X\text{-Mod}$*, we have*

$$\mathrm{endol}\big(F_X(N)\big) \leq \mu(X) \times \mathrm{endol}(N).$$

2. *For any k-algebra E, the induced functor $F_X^E : \mathcal{A}^X\text{-}E\text{-Mod} \longrightarrow \mathcal{A}\text{-}E\text{-Mod}$ satisfies that $\ell_E(F_X^E(N)) \leq \mu(X) \times \ell_E(N)$, for any $N \in \mathcal{A}^X\text{-}E\text{-Mod}$.*

Proof. The functor F_X is full and faithful by [6], (13.5).

Given a k-algebra E and $N \in \mathcal{A}^X\text{-}E\text{-Mod}$, as in the proof of 4.4, see also [6], (25.7), we can show that, in order to calculate the length of the E-module $F_X^E(N)$, we can calculate the length of the E-module $X \otimes_S N$ with the usual action of E on the second tensor factor N. From an epimorphism $S^{\mu(X)} \longrightarrow X$ of right S-modules, we obtain an epimorphism of right E-modules $S^{\mu(X)} \otimes_S N \longrightarrow X \otimes_S N$. Then, we obtain: $\ell_E(F_X^E(N)) = \ell_E(X \otimes_S N) \leq \ell_E(N^{\mu(X)}) = \mu(X) \times \ell_E(N)$. Then, (2) holds. Item (1) is obtained from (2) as before: take $E = \mathrm{End}_{\mathcal{A}^X}(N)^{op}$ and notice that $\mathrm{endol}(N) = \ell_E(N)$ and $\mathrm{endol}(F_X(N)) = \ell_E(F_X^E(N))$, because F_X is full and faithful. □

We need to adapt to our context some definitions and results due to Crawley-Boevey, see [9], §5. Given a principal ideal k-domain Γ (which we always assume to be infinite-dimensional over k), we denote by $\mathrm{Irred}(\Gamma)$ a complete set of inequivalent irreducible elements of Γ. The following definition is a little bit more general than [5], (2.4). It generalizes the original definition of realization given by Crawley-Boevey in [9] for generic modules over finite-dimensional algebras.

Definition 4.8. Let $\mathcal{A}$ be a layered ditalgebra over any field k, G a pregeneric $\mathcal{A}$-module, and Γ some principal ideal k-domain. Then, a *realization Z for G over Γ* is an A-Γ-bimodule Z, which is finitely generated as a right Γ-module and such that

$$G \cong Z \otimes_\Gamma Q \quad \text{in } \mathcal{A}\text{-Mod} \quad \text{and} \quad \mathrm{endol}(G) = \dim_Q(Z \otimes_\Gamma Q),$$

where Q denotes the skew field of fractions of Γ.

Remark 4.9. Assume that G and G' are isomorphic pregeneric $\mathcal{A}$-modules. Then, if Z is a realization of G over Γ, it is also a realization of G' over Γ.

The precedent definition has an obvious version for an arbitrary algebra Λ (maybe infinite-dimensional) and a pregeneric Λ-module G (and we have the remark corresponding to the previous statement). If Λ is finite-dimensional, the pregeneric Λ-modules coincide with the generic Λ-modules.

Remark 4.10. If Γ is a principal ideal k-domain and Q is its skew field of fractions, then Γ is a realization of the pregeneric Γ-module Q. In this paper, realizations of pregeneric modules will appear as images of realizations of pregeneric modules under compositions of the special kind functors studied in Lemmas 4.14 and 4.16–4.19, starting from the case of a principal ideal k-domain just described.

Lemma 4.11. *Let $\mathcal{A}$ be a layered ditalgebra, Γ a principal ideal k-domain with skew field of fractions Q, and Z an A-Γ-bimodule, finitely generated by the right, such that* *the $\mathcal{A}$-module $G = Z \otimes_\Gamma Q$ is pregeneric. Make $E_G = \operatorname{End}_{\mathcal{A}}(G)^{op}$ and assume that $Q_G := E_G/\operatorname{rad} E_G$ is a skew field. Let Q_Z be the k-subalgebra of E_G defined as the image of the k-algebra morphism $\mu : Q \longrightarrow E_G$ such that $\mu(q) = (id_Z \otimes \mu_q, 0)$, where $q \in Q$ and μ_q is right multiplication by q in Q. Then, Z is a realization of G iff $\eta(Q_Z) = Q_G$, where $\eta = \eta_G : E_G \longrightarrow E_G/\operatorname{rad} E_G = Q_G$ denotes the quotient map.*

Proof. Let $0 \subseteq G_1 \subseteq \cdots \subseteq G_\ell = G$ be a composition series of the E_G-module G. Since $Q_Z \leq E_G$ this is a series of Q_Z-subspaces of G. We have that $\eta(Q_Z) \leq Q_G$, each G_i/G_{i-1} is a one-dimensional Q_G-vector space, so each G_i/G_{i-1} is a Q_Z-vector space with $\dim_{Q_Z}(G_i/G_{i-1}) = \dim_{\eta(Q_Z)}(G_i/G_{i-1})$. It follows that, $[Q_G : \eta(Q_Z)] = \dim_{Q_Z}(G_i/G_{i-1})$, and we obtain

$$\dim_Q(Z \otimes_\Gamma Q) = \dim_{Q_Z} G = \operatorname{endol}(G)\big[Q_G : \eta(Q_Z)\big].$$

As a consequence, Z is a realization of G over Γ iff $\eta(Q_Z) = Q_G$. □

We have the corresponding statement for finite-dimensional algebras with a similar proof.

Lemma 4.12. *Let Λ be a finite-dimensional algebra, Γ a principal ideal k-domain with skew field of fractions Q, and Z a Λ-Γ-bimodule, finitely generated by the right, such that $G = Z \otimes_\Gamma Q$ is a generic Λ-module. Denote by $E_G = \operatorname{End}_\Lambda(G)^{op}$ and assume that $Q_G := E_G/\operatorname{rad} E_G$ is a skew field. Let Q_Z be the k-subalgebra of E_G defined as the image of the k-algebra morphism $\mu : Q \longrightarrow E_G$ such that $\mu(q) = id_Z \otimes \mu_q$, where $q \in Q$ and μ_q is right multiplication by q in Q. Then, Z is a realization of G iff $\eta(Q_Z) = Q_G$, where $\eta = \eta_G : E_G \longrightarrow E_G/\operatorname{rad} E_G = Q_G$ denotes the quotient map.*

In the context of the last lemma, if k is algebraically closed, Λ is tame, and G is a generic Λ-module, by Crawley-Boevey's work, we know that $Q_G \cong k(x)$. If $\Gamma = k[x]_f$ is a rational algebra with field of fractions $Q = k(x)$, then the last lemma applies to any generic Λ-module of the form $G = Z \otimes_\Gamma Q$.

Lemma 4.13. *Assume that $F : \mathcal{A}\text{-Mod} \longrightarrow \mathcal{B}\text{-Mod}$ is a k-linear functor, where $\mathcal{A}$ and $\mathcal{B}$ are layered ditalgebras. Suppose that Γ is a principal ideal k-domain with skew field of fractions Q, and Z is an A-Γ-bimodule. Assume that F satisfies the following:*

1. *$F(Z)$ admits a structure of right Γ-module, transforming $F(Z)$ into a B-Γ-bimodule.*
2. *There is an isomorphism $\rho : F(Z \otimes_\Gamma Q) \longrightarrow F(Z) \otimes_\Gamma Q$ of $\mathcal{B}$-modules such that, for any $q \in Q$, the following square commutes in $\mathcal{B}$-Mod*

$$\begin{array}{ccc} F(Z \otimes_\Gamma Q) & \xrightarrow{\rho} & F(Z) \otimes_\Gamma Q \\ \downarrow{\scriptstyle F(1_Z \otimes \mu_q, 0)} & & \downarrow{\scriptstyle (1_{F(Z)} \otimes \mu_q, 0)} \\ F(Z \otimes_\Gamma Q) & \xrightarrow{\rho} & F(Z) \otimes_\Gamma Q \end{array}$$

3. *The morphism* $\phi : E_{Z\otimes_\Gamma Q} \longrightarrow E_{F(Z\otimes_\Gamma Q)}$ *given by* F *maps* $\operatorname{rad} E_{Z\otimes_\Gamma Q}$ *into* $\operatorname{rad} E_{F(Z\otimes_\Gamma Q)}$ *and so it induces a morphism* $\underline{\phi}$ *in the commutative square*

$$\begin{array}{ccc} E_{Z\otimes_\Gamma Q} & \xrightarrow{\phi} & E_{F(Z\otimes_\Gamma Q)} \\ {\scriptstyle \eta_{Z\otimes_\Gamma Q}}\downarrow & & \downarrow{\scriptstyle \eta_{F(Z\otimes_\Gamma Q)}} \\ Q_{Z\otimes_\Gamma Q} & \xrightarrow{\underline{\phi}} & Q_{F(Z\otimes_\Gamma Q)} \end{array}$$

where $Q_{Z\otimes_\Gamma Q}$ *and* $Q_{F(Z\otimes_\Gamma Q)}$ *denote the quotient algebras of* $E_{Z\otimes_\Gamma Q}$ *and* $E_{F(Z\otimes_\Gamma Q)}$ *modulo the radical, respectively, and the vertical maps are the canonical projections.*

As before, denote by Q_Z *(resp.* $Q_{F(Z)}$*) the subalgebra of* $E_{Z\otimes_\Gamma Q}$ *(resp.* $E_{F(Z)\otimes_\Gamma Q}$*) determined by the morphisms of the form* $(1\otimes\mu_q, 0)$*, where* μ_q *is multiplication by* q *and* q *runs in* Q*. Then,*

a. *If* $\underline{\phi}$ *is injective and* $\eta_{F(Z)\otimes_\Gamma Q}(Q_{F(Z)}) = Q_{F(Z)\otimes_\Gamma Q}$*, then* $\eta_{Z\otimes_\Gamma Q}(Q_Z) = Q_{Z\otimes_\Gamma Q}$*;*
b. *If* $\underline{\phi}$ *is a bijective map and* $\eta_{Z\otimes_\Gamma Q}(Q_Z) = Q_{Z\otimes_\Gamma Q}$*, then* $\eta_{F(Z)\otimes_\Gamma Q}(Q_{F(Z)}) = Q_{F(Z)\otimes_\Gamma Q}$.

Proof. Let us denote by ψ the following composition of morphisms of algebras:

$$E_{Z\otimes_\Gamma Q} \xrightarrow{\phi} E_{F(Z\otimes_\Gamma Q)} \xrightarrow{\zeta} E_{F(Z)\otimes_\Gamma Q},$$

where ζ is the isomorphism given by conjugation by ρ. From our assumption in item *2*, we obtain $\psi(Q_Z) = Q_{F(Z)}$. Then, from our assumption in item *3*, we get $\eta_{F(Z)\otimes_\Gamma Q}(Q_{F(Z)}) = \eta_{F(Z)\otimes_\Gamma Q}\psi(Q_Z) = \underline{\psi}\eta_{Z\otimes_\Gamma Q}(Q_Z)$, where, again, $\eta_{F(Z)\otimes_\Gamma Q} : E_{F(Z)\otimes_\Gamma Q} \longrightarrow Q_{F(Z)\otimes_\Gamma Q}$ denotes the canonical projection to the quotient algebra of $E_{F(Z)\otimes_\Gamma Q}$ modulo its radical, and $\underline{\psi} : Q_{Z\otimes_\Gamma Q} \longrightarrow Q_{F(Z)\otimes_\Gamma Q}$ is the map induced by ψ. Then, under the assumptions of a, the map $\underline{\psi}$ is injective and $\underline{\psi}(Q_{Z\otimes_\Gamma Q}) \subseteq Q_{F(Z)\otimes_\Gamma Q} = \eta_{F(Z)\otimes_\Gamma Q}(Q_{F(Z)}) = \underline{\psi}\eta_{Z\otimes_\Gamma Q}(Q_Z)$. Thus, $\eta_{Z\otimes_\Gamma Q}(Q_Z) = Q_{Z\otimes_\Gamma Q}$. While, under the assumptions of b, the map $\underline{\psi}$ is bijective and $\eta_{F(Z)\otimes_\Gamma Q}(Q_{F(Z)}) = \underline{\psi}\eta_{Z\otimes_\Gamma Q}(Q_Z) = \underline{\psi}(Q_{Z\otimes_\Gamma Q}) = Q_{F(Z)\otimes_\Gamma Q}$. □

Lemma 4.14. *Let* Λ *be a finite-dimensional algebra over a perfect field* k *and consider its Drozd's ditalgebra* $\mathcal{D}$*. Consider the usual equivalence functor* $\Xi_\Lambda : \mathcal{D}\text{-Mod} \longrightarrow \mathcal{P}^1(\Lambda)$*, the cokernel functor* $\operatorname{Cok} : \mathcal{P}^1(\Lambda) \longrightarrow \Lambda\text{-Mod}$*, and the transition bimodule* Z*, as in [6], (22.18). Let* Γ *be a principal ideal* k*-domain with skew field of fractions* Q*. Assume that* G *is a pregeneric* $\mathcal{D}$*-module with realization* Z_0 *over* Γ*, and that* Q_G *is a skew field. Then,* $Z\otimes_D Z_0$ *is a realization of the generic* Λ*-module* $\operatorname{Cok}\Xi_\Lambda(G)$ *over* Γ*.*

Proof. Denote by F the following composition of functors

$$D\text{-Mod} \xrightarrow{L_{\mathcal{D}}} \mathcal{D}\text{-Mod} \xrightarrow{\Xi_\Lambda} \mathcal{P}^1(\Lambda) \xrightarrow{\operatorname{Cok}} \Lambda\text{-Mod}.$$

By [6], (22.18), there is an isomorphism of functors $F(Z_0 \otimes_\Gamma -) \longrightarrow F(Z_0) \otimes_\Gamma -$. In particular, we have an isomorphism $\rho : F(Z_0 \otimes_\Gamma Q) \longrightarrow F(Z_0) \otimes_\Gamma Q$ such that $(1_{F(Z_0)\otimes_\Gamma Q} \otimes \mu_q, 0)\rho = \rho F(1_{Z_0} \otimes \mu_q, 0)$, for all $q \in Q$. Thus, items *1* and *2* of 4.13 are satisfied by F. By [6], (18.10), the map $\underline{\phi}$ of 4.13 is well defined here and surjective. It is injective because $Q_G \cong Q_{Z_0 \otimes_\Gamma Q}$ is a skew field. By assumption, Z_0 is a realization of $Z_0 \otimes_\Gamma Q$, thus $\eta_{Z_0 \otimes_\Gamma Q}(Q_{Z_0}) = Q_{Z_0 \otimes_\Gamma Q}$, then from 4.13($b$), $\eta_{F(Z_0)\otimes_\Gamma Q}(Q_{F(Z_0)}) = Q_{F(Z_0)\otimes_\Gamma Q}$. We also know that $\mathrm{Cok}\, \Xi_\Lambda(G) = F(G) \cong F(Z_0 \otimes_\Gamma Q) \cong F(Z_0) \otimes_\Gamma Q$ is a generic Λ-module. Thus, $F(Z_0)$ is a realization of $F(Z_0) \otimes_\Gamma Q$ over Γ. By [6], (22.18)(2), there is an isomorphism of functors $F \cong Z \otimes_D -$. Moreover, $\mathrm{Cok}\, \Xi_\Lambda(Z_0) = F(Z_0) \cong Z \otimes_D Z_0$ as Λ-Γ-bimodules, so $Z \otimes_D Z_0$ is a realization of $\mathrm{Cok}\, \Xi_\Lambda(G)$ over Γ. □

Lemma 4.15. *Let Λ be a tame finite-dimensional algebra over an algebraically closed field k and consider its Drozd's ditalgebra $\mathcal{D}$. For $i \in [1, 2]$, let G_i be a pregeneric $\mathcal{D}$-module, Z_i a D-Γ_i-bimodule, where Γ_i is a rational k-algebra and Z_i is a finitely generated right Γ_i-module. Assume that Z_1 and Z_2 are realizations of G_1 and G_2, over Γ_1 and Γ_2, respectively. If there is an infinite subset P of* Irred(Γ_2) *such that, for all $p \in P$, we have*

$$Z_2 \otimes_{\Gamma_2} \Gamma_2/\big(p^{i_p}\big) \cong Z_1 \otimes_{\Gamma_1} \Gamma_1/(q_p) \quad \text{in } \mathcal{D}\text{-Mod},$$

for some $q_p \in \Gamma_1$ and $i_p \in \mathbb{N}$, then $G_2 \cong G_1$.

Proof. The proof given in [5], (2.11) also works here for our possibly non-basic finite-dimensional algebra Λ. We have to use 4.14 and Crawley-Boevey's results on tame algebras [9], (4.4) and [9], (5.2)(4). Here Drozd's ditalgebra $\mathcal{D}$ may be not admissible (hence may be not seminested). □

Lemma 4.16. *Let $\mathcal{A}_0$ be a proper subditalgebra of the layered ditalgebra $\mathcal{A}$. Denote by $E_0 : \mathcal{A}_0\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$ the extension functor, as in 4.1. Assume that $M \cong N$ in $\mathcal{A}_0$-Mod implies that $E_0(M) \cong E_0(N)$ in $\mathcal{A}$-Mod. Suppose that Z_0 is a realization of a pregeneric $\mathcal{A}_0$-module G_0 over a principal ideal k-domain Γ. Then, $E_0(G_0)$ is a pregeneric $\mathcal{A}$-module. If we have furthermore that E_{G_0} and $E_{E_0(G_0)}$ are local with nilpotent radical, then $E_0(Z_0)$ is a realization of the pregeneric $\mathcal{A}$-module $E_0(G_0)$ over Γ.*

Proof. We know that Z_0 is a realization of $Z_0 \otimes_\Gamma Q$, where Q is the skew field of fractions of Γ. In particular, $G_0 \cong Z_0 \otimes_\Gamma Q$ in $\mathcal{A}_0$-Mod and, by assumption, $E_0(Z_0) \otimes_\Gamma Q \cong E_0(Z_0 \otimes_\Gamma Q) \cong E_0(G_0)$ in $\mathcal{A}$-Mod. By [4], (2.4)(1), $E_0(G_0)$ is indecomposable in $\mathcal{A}$-Mod. As in [5], (2.5), we can show that $\mathrm{endol}(E_0(Z_0) \otimes_\Gamma Q)$ is finite. It follows that $E_0(G_0)$ is a pregeneric $\mathcal{A}$-module.

It is clear that $FL_{\mathcal{A}}(E_0(Z_0) \otimes_\Gamma -) = L_{\mathcal{A}_0}(F(E_0(Z_0)) \otimes_\Gamma -)$, where $F = R_0 : \mathcal{A}\text{-Mod} \longrightarrow \mathcal{A}_0\text{-Mod}$ denotes the restriction functor. Thus, items *1* and *2* of 4.13 are satisfied by this functor F and for the A-Γ-bimodule $Z = E_0(Z_0)$.

Now, assume that E_{G_0} and $E_{E_0(G_0)}$ are local with nilpotent radical. Then, the morphism $\phi : E_{E_0(G_0)} \longrightarrow E_{R_0(E_0(G_0))} = E_{G_0}$ given by the restriction functor induces an injective map modulo the radicals: $\underline{\phi} : Q_{E_0(G_0)} \longrightarrow Q_{R_0(E_0(G_0))} = Q_{G_0}$. Since $R_0(E_0(Z_0)) = Z_0$ is a realization of $R_0(E_0(Z_0)) \otimes_\Gamma Q = Z_0 \otimes_\Gamma Q$ over Γ, by 4.11, we get $\eta_{R_0(E_0(Z_0))\otimes_\Gamma Q}(Q_{R_0(E_0(Z_0))}) = Q_{R_0(E_0(Z_0))\otimes_\Gamma Q}$. Then, by 4.13$(a)$, we get $\eta_{E_0(Z_0)\otimes_\Gamma Q}(Q_{E_0(Z_0)}) = Q_{E_0(Z_0)\otimes_\Gamma Q}$. Again by 4.11, this means that $E_0(Z_0)$ is a realization for the pregeneric $\mathcal{A}$-module $E_0(Z_0) \otimes_\Gamma Q$ over Γ. □

Lemma 4.17. *Assume that $\xi : \mathcal{A} \longrightarrow \mathcal{A}'$ is a morphism of layered ditalgebras and consider the functor $F_\xi : \mathcal{A}'$-Mod $\longrightarrow \mathcal{A}$-Mod induced by restriction using the morphism ξ. Let Γ be a principal ideal k-domain with skew field of fractions Q. Assume that G is a pregeneric $\mathcal{A}'$-module with realization Z over Γ and that Q_G is a skew field. Then if F_ξ is full and faithful, the pregeneric $\mathcal{A}$-module $F_\xi(G)$ admits the realization $F_\xi(Z)$ over Γ.*

Proof. There is an isomorphism $FL_{\mathcal{A}'}(Z \otimes_\Gamma -) \longrightarrow L_{\mathcal{A}}(F(Z) \otimes_\Gamma -)$, where $F = F_\xi$, and $L_{\mathcal{A}'}$ and $L_{\mathcal{A}}$ denote canonical embeddings. In particular, we have an isomorphism $\rho : F(Z \otimes_\Gamma Q) \longrightarrow F(Z) \otimes_\Gamma Q$ satisfying item *2* of 4.13. Since F_ξ is full and faithful, the map $\underline{\phi}$ of 4.13 is well defined here and bijective. By assumption, Z is a realization of $Z \otimes_\Gamma Q$; by 4.11, $\eta_{Z\otimes_\Gamma Q}(Q_Z) = Q_{Z\otimes_\Gamma Q}$; then from 4.13$(b)$, $\eta_{F(Z)\otimes_\Gamma Q}(Q_{F(Z)}) = Q_{F(Z)\otimes_\Gamma Q}$. Then, $F(Z)$ is a realization of $F(Z) \otimes_\Gamma Q$ over Γ, so $F(Z)$ is a realization of $F(G)$ over Γ. □

Lemma 4.18. *Let $\mathcal{A}$ be a layered ditalgebra over a field k. Assume that $\mathcal{A}^X$ is the layered ditalgebra obtained from $\mathcal{A}$ by reduction, using a complete triangular admissible $\mathcal{A}'$-module X, where $\mathcal{A}'$ is a proper subalgebra of $\mathcal{A}$. By [6], (13.5), the associated functor $F^X : \mathcal{A}^X$-Mod $\longrightarrow \mathcal{A}$-Mod is full and faithful. Assume that Z is a realization of a pregeneric $\mathcal{A}^X$-module G over a principal ideal k-domain Γ, where Q_G is a skew field. Then $F^X(Z)$ is a realization of the pregeneric $\mathcal{A}$-module $F^X(G)$ over the algebra Γ.*

Proof. Here we also have an isomorphism $FL_{\mathcal{A}^X}(Z \otimes_\Gamma -) \longrightarrow L_{\mathcal{A}}(F(Z) \otimes_\Gamma -)$, where $F = F^X$, and $L_{\mathcal{A}^X}$ and $L_{\mathcal{A}}$ denote canonical embeddings. Then, we can proceed as in the precedent proof. □

With a similar argument we can show the following.

Lemma 4.19. *Let A and B be k-algebras, and $F : B$-Mod $\longrightarrow A$-Mod a full and faithful functor of the form $F \cong Y \otimes_B -$, where Y is an A-B-bimodule, which is finitely generated by the right. Let Z be a realization of a pregeneric B-module G over a principal ideal k-domain Γ, where Q_G is a skew field. Then $F(Z)$ is a realization of the pregeneric A-module $F(G)$ over the algebra Γ.*

Remark 4.20. Recall that given a seminested ditalgebra $\mathcal{A}$, over an algebraically closed field, there are some basic types of ditalgebra operations $\mathcal{A} \mapsto \mathcal{A}^z$, where $z \in \{a, r, d, e, u\}$. Namely: absorption of a loop $\mathcal{A} \mapsto \mathcal{A}^a$, as in [6], (23.16); regularization $\mathcal{A} \mapsto \mathcal{A}^r$, as in [6], (23.15); deletion of idempotents $\mathcal{A} \mapsto \mathcal{A}^d$, as in [6], (23.14); edge reduction $\mathcal{A} \mapsto \mathcal{A}^e$, as in [6], (23.18); and unravelling $\mathcal{A} \mapsto \mathcal{A}^u$, as in [6], (23.23).

Associated to each one of these operations $\mathcal{A} \mapsto \mathcal{A}^z$, there is an associated basic reduction functor $F^z : \mathcal{A}^z\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$.

The functors F^a, F^r, and F^d are full and faithful, by [6], (8.20), [6], (8.19), and [6], (8.17), respectively. The functors F^e and F^u are full and faithful because they are of type F^X, where X is a complete admissible module, by [6], (17.12). Then, we can derive from the preceding Lemmas 4.18 and 4.17 that any composition of functors of these types preserves realizations of pregeneric modules G such Q_G is a skew field.

Corrigendum 4.21. There are some inaccuracies in [5] which need to be pointed out and corrected. The problem arises in [5], (2.9), which is an incorrect statement, see Example 4.22. This can be replaced by Lemma 4.11, which provides a precise statement which can be particularized to the context of [5]: that is the case of an algebraically closed field k, a tame seminested ditalgebra $\mathcal{A}$, and a pregeneric $\mathcal{A}$-module of the form $G = Z \otimes_\Gamma k(x)$, where Γ is a rational k-algebra. Recall that in this case, $E_G / \operatorname{rad} E_G \cong k(x)$, according to [5], (2.7). The other numbered statements of [5] are correct, but we need to make some adjustments to some of their proofs. Namely:

1. In the proof of [5], (2.10)(i), we need to use that any composition of basic reduction functors preserves realizations: 4.20.
2. The proposition [5], (2.11) can be improved to our Lemma 4.15, with the same proof using 4.14, as we mentioned above.
3. In the proof of [5], (3.4) we have to replace the first use of [5], (2.9) by the use of 4.16, and the second use of [5], (2.9) by the use of 4.18.

Thus, with the only exception of [5], (2.9), we are free to use the results stated in [5].

The following example shows that in the definition of realization of a generic module, the requirement $\operatorname{endol}(G) = \dim_Q(Z \otimes_\Gamma Q)$ does not follow from the fact that $G \cong Z \otimes_\Gamma Q$.

Example 4.22. Consider the Kronecker k-algebra Λ, that is the path k-algebra of the quiver $\cdot \rightrightarrows \cdot$ and a natural number $n \geq 2$. Consider the rational algebra $\Gamma = k[x^n] \subseteq k[x]$ and its field of fractions $Q = k(x^n) \subseteq k(x)$. Then, consider the Λ-Γ-bimodule Z corresponding to the representation

$$k[x] \underset{1}{\overset{x}{\rightrightarrows}} k[x],$$

where Γ acts on $k[x]$ through the inclusion map. Since $k[x] = \bigoplus_{i=0}^{n-1} x^i\Gamma$ and $k(x) = \bigoplus_{i=0}^{n-1} x^iQ$, the product map $k[x] \otimes_\Gamma Q \longrightarrow k(x)$ is an isomorphism of $k[x]$-modules. Thus, the Λ-module $G = Z \otimes_\Gamma Q$ corresponds to the representation

$$k(x) \underset{1}{\overset{x}{\rightrightarrows}} k(x).$$

Thus, G is a generic Λ-module with $\mathrm{endol}(G) = 2$, while $\dim_Q(Z \otimes_\Gamma Q) = 2n$.

5. Restrictions and endolength

In this section we prove Theorem 5.4, which plays an essential role in the proof of our main result. For this, we need an improved version of 4.15, namely Lemma 5.2.

Definition 5.1. We say that a seminested ditalgebra $\mathcal{A}_0$, over an algebraically closed field k, is *biconstructible* iff there is an admissible ditalgebra $\mathcal{A}$, which is constructible from a finite-dimensional algebra Λ, and a finite sequence of reductions

$$\mathcal{A} \longmapsto \mathcal{A}^{x_1} \longmapsto \mathcal{A}^{x_1x_2} \longmapsto \cdots \longmapsto \mathcal{A}^{x_1\cdots x_s} = \mathcal{A}_0,$$

for some finite set of ditalgebra operations $\mathcal{A}^{x_1\cdots x_{j-1}} \longmapsto \mathcal{A}^{x_1\cdots x_j}$ of either of the types: absorption of a loop, regularization, deletion of idempotents, edge reduction, and unravelling. In this case, we say that $\mathcal{A}_0$ is *biconstructible from Λ, through $\mathcal{A}$.*

From [4], (3.4), the ditalgebra $\mathcal{A}$ is seminested.

Lemma 5.2. *Let $\mathcal{A}_0$ be a seminested ditalgebra, which is biconstructible from a tame finite-dimensional algebra, over an algebraically closed field k. Let Z_1 and Z_2 be realizations of some pregeneric $\mathcal{A}_0$-modules G_1 and G_2, over some rational algebras Γ_1 and Γ_2, respectively. If there is an infinite subset P of* $\mathrm{Irred}(\Gamma_2)$ *such that, for all $p \in P$, we have*

$$Z_2 \otimes_{\Gamma_2} \Gamma_2/\big(p^{i_p}\big) \cong Z_1 \otimes_{\Gamma_1} \Gamma_1/(q_p) \quad \text{in } \mathcal{A}_0\text{-Mod},$$

for some $q_p \in \Gamma_1$ and $i_p \in \mathbb{N}$, then $G_2 \cong G_1$.

Proof. Assume that the ditalgebra $\mathcal{A}_0$ is biconstructible from a tame algebra Λ, through the admissible ditalgebra $\mathcal{A}$. Adopt the notation of 2.3 and 5.1.

Consider an isomorphism of layered ditalgebras $\xi : \mathcal{D}^{z_1\cdots z_t} \longrightarrow \mathcal{A}$ and the corresponding restriction functor $F_\xi : \mathcal{A}\text{-Mod} \longrightarrow \mathcal{D}^{z_1\cdots z_t}\text{-Mod}$. For $i \in [1,t]$, consider the functor $F_i : \mathcal{D}^{z_1\cdots z_i}\text{-Mod} \longrightarrow \mathcal{D}^{z_1\cdots z_{i-1}}\text{-Mod}$ associated to the corresponding reduction $\mathcal{D}^{z_1\cdots z_{i-1}} \longmapsto \mathcal{D}^{z_1\cdots z_i}$. For $j \in [1,s]$, consider the functor $H_j : \mathcal{A}^{x_1\cdots x_j}\text{-Mod} \longrightarrow \mathcal{A}^{x_1\cdots x_{j-1}}\text{-Mod}$ associated to the corresponding operation $\mathcal{A}^{x_1\cdots x_{j-1}} \longmapsto \mathcal{A}^{x_1\cdots x_j}$. Then, the composition

$$F := F_1F_2\cdots F_tF_\xi H_1H_2\cdots H_s : \mathcal{A}_0\text{-Mod} \longrightarrow \mathcal{D}\text{-Mod}$$

is a full and faithful functor which preserves pregeneric modules, by [4], (2.5)–(2.6) and 4.7(1). From [6], (22.7), there is a D-A_0-bimodule Z, finitely generated as a right A_0-module, such that $F(M) \cong Z \otimes_{A_0} M$, for $M \in \mathcal{A}_0$-Mod. Hence, $F(G_i) \cong Z \otimes_{A_0} Z_i \otimes_{\Gamma_i} k(x)$, and $Z \otimes_{A_0} Z_i$ is a realization of the pregeneric $\mathcal{D}$-module $F(G_i)$ over the rational algebra Γ_i, for $i \in [1,2]$, see 4.18 and 4.17. Moreover,

$$\begin{aligned} Z \otimes_{A_0} Z_2 \otimes_{\Gamma_2} \Gamma_2/\big(p^{i_p}\big) &\cong F\big(Z_2 \otimes_{\Gamma_2} \Gamma_2/\big(p^{i_p}\big)\big) \\ &\cong F\big(Z_1 \otimes_{\Gamma_1} \Gamma_1/(q_p)\big) \\ &\cong Z \otimes_{A_0} Z_1 \otimes_{\Gamma_1} \Gamma_1/(q_p). \end{aligned}$$

Then, from 4.15, we obtain that $F(G_1) \cong F(G_2)$. Since F is full and faithful, we also get $G_1 \cong G_2$, as claimed. □

Proposition 5.3. *Assume that $\mathcal{A}'_0$ is an initial subditalgebra of a seminested ditalgebra $\mathcal{A}_0$, which is biconstructible from a tame algebra Λ, over an algebraically closed field k. Consider the extension functor $E_0 : \mathcal{A}'_0\text{-Mod} \longrightarrow \mathcal{A}_0\text{-Mod}$ and the restriction functor $R_0 : \mathcal{A}_0\text{-Mod} \longrightarrow \mathcal{A}'_0\text{-Mod}$. Assume furthermore that $E_0(M) \cong E_0(N)$ in $\mathcal{A}_0$-Mod whenever $M \cong N$ in $\mathcal{A}'_0$-Mod. Then, for any $d \in \mathbb{N}$, there is a finite family $\mathcal{I}(d)$ of finite-dimensional indecomposable $\mathcal{A}'_0$-modules such that:*

1. *For any indecomposable $\mathcal{A}_0$-module M with $\dim_k M \leq d$ and $M \ncong E_0(N)$ in $\mathcal{A}_0$-Mod, for any $N \in \mathcal{A}'_0$-Mod, the module $R_0(M)$ is isomorphic in $\mathcal{A}'_0$-Mod to a direct sum of modules in $\mathcal{I}(d)$;*
2. *For any pregeneric $\mathcal{A}_0$-module G with $\operatorname{endol}(G) \leq d$ and $G \ncong E_0(H)$ in $\mathcal{A}_0$-Mod, for any pregeneric $\mathcal{A}'_0$-module H, the module $R_0(G)$ is isomorphic in $\mathcal{A}'_0$-Mod to a direct sum of modules in $\mathcal{I}(d)$.*

Proof. Essentially the same argument given in the proof of [5], (3.4) works here, but we have to be careful because Λ may be non-basic. We give the details.

By assumption, $\mathcal{A}_0$ is biconstructible from a tame algebra Λ, through an admissible ditalgebra $\mathcal{A}$. Since $\mathcal{A}$ is constructible from a tame algebra, it is pregenerically tame, see [4], (4.6). We claim that $\mathcal{A}_0$ is tame. Indeed, if this was not the case, by Drozd's Theorem, the ditalgebra $\mathcal{A}_0$ is wild, see [6], (27.10). From [6], (22.8) and [6], (22.10), we obtain that $\mathcal{A}$ is wild. Hence, from [4], (2.9), we would have that $\mathcal{A}$ is not pregenerically tame, which is not the case. Thus, $\mathcal{A}_0$ is indeed tame. It follows that $\mathcal{A}'_0$ is also tame, see [6], (22.13).

Fix $d \in \mathbb{N}$ and apply [3], (4.1) to $\mathcal{A}_0$ and $\mathcal{A}'_0$, to obtain a finite set $\mathcal{I}(d) := \{X_1, \ldots, X_t\}$ of pairwise non-isomorphic finite-dimensional indecomposable $\mathcal{A}'_0$-modules satisfying the first item.

In order to prove the second item, take a pregeneric $\mathcal{A}_0$-module G such that $\operatorname{endol}(G) \leq d$ and $G \ncong E_0(H)$, for any pregeneric $\mathcal{A}'_0$-module H.

From [5], (2.10), there is a realization Z of G over a rational algebra Γ, which is free finitely generated as a right Γ-module. It defines the infinite family of pairwise non-isomorphic indecomposable $\mathcal{A}_0$-modules

$$\{Z \otimes_\Gamma \Gamma/(p) \mid p \in \mathrm{Irred}(\Gamma)\}.$$

If $\mathrm{rk}(Z)$ denotes the rank of Z as a free right Γ-module, then $\mathrm{rk}(Z) = \dim_{k(x)}(Z \otimes_\Gamma k(x)) = \mathrm{endol}(G) \leq d$ and, for each $p \in \mathrm{Irred}(\Gamma)$, we have that

$$\dim_k Z \otimes_\Gamma \Gamma/(p) \leq d.$$

Then, for any $p \in \mathrm{Irred}(\Gamma)$ with $Z \otimes_\Gamma \Gamma/(p) \ncong E_0(N)$ in $\mathcal{A}_0$-Mod, for any $N \in \mathcal{A}'_0$-Mod, the module $R_0(Z \otimes_\Gamma \Gamma/(p))$ is isomorphic in $\mathcal{A}'_0$-Mod to a direct sum of modules in $\mathcal{I}(d)$. Since $\mathcal{A}'_0$ is a Roiter ditalgebra and k is algebraically closed, from [5], (3.3), we have the admissible $\mathcal{A}'_0$-module $X := \bigoplus_{i=1}^t X_i$, the admissible seminested ditalgebra $\mathcal{A}_0^X$, see [4], (3.4), and the associated reduction functor $F^X : \mathcal{A}_0^X\text{-Mod} \longrightarrow \mathcal{A}_0\text{-Mod}$. The functor F^X is full and faithful by [6], (13.3) and [6], (13.5). Let us first prove the following.

Claim. *There is no infinite subset P of* $\mathrm{Irred}(\Gamma)$ *such that, for all $p \in P$, there is $N_p \in \mathcal{A}'_0$*-Mod *with $Z \otimes_\Gamma \Gamma/(p) \cong E_0(N_p)$.*

Proof of Claim. Assume that there is such a set P. Then, the seminested tame ditalgebra $\mathcal{A}'_0$ admits the infinite family $\{N_p\}_{p\in P}$ of pairwise non-isomorphic indecomposable $\mathcal{A}'_0$-modules with $\dim_k N_p \leq d$. Then, from [5], (2.10), there are a pregeneric $\mathcal{A}'_0$-module G', a realization Z' of G' over a rational algebra Γ', and an infinite subset Q of $\mathrm{Irred}(\Gamma')$ such that, for any $q \in Q$, there are $p_q \in P$ and $i_q \in \mathbb{N}$ with

$$Z' \otimes_{\Gamma'} \Gamma'/\bigl(q^{i_q}\bigr) \cong N_{p_q} \quad \text{in } \mathcal{A}'_0\text{-Mod}.$$

Then, for all $q \in Q$, we have

$$Z \otimes_\Gamma \Gamma/(p_q) \cong E_0(N_{p_q}) \cong E_0\bigl(Z' \otimes_{\Gamma'} \Gamma'/\bigl(q^{i_q}\bigr)\bigr) \cong E_0\bigl(Z'\bigr) \otimes_{\Gamma'} \Gamma'/\bigl(q^{i_q}\bigr).$$

Moreover, $E_0(G') \cong E_0(Z' \otimes_{\Gamma'} k(x)) \cong E_0(Z') \otimes_{\Gamma'} k(x)$. From 4.16, we have that $E_0(G')$ is a pregeneric $\mathcal{A}_0$-module with realization $E_0(Z')$ over Γ'. Then, from 5.2, we obtain that $E_0(G') \cong G$, contradicting our initial assumption. This ends the proof of our Claim. □

Then, there are infinitely many elements $p \in \mathrm{Irred}(\Gamma)$ such that

$$Z \otimes_\Gamma \Gamma/(p) \ncong E_0(N), \quad \text{for any } N \in \mathcal{A}'_0\text{-Mod}.$$

Hence, there is an infinite subset $P \subseteq \mathrm{Irred}(\Gamma)$ such that, for any $p \in P$, the module $R_0(Z \otimes_\Gamma \Gamma/(p))$ is isomorphic in $\mathcal{A}'_0$-Mod to a direct sum of direct summands of X.

From [5], (3.3), we know that, for each $p \in P$, there is an $\mathcal{A}_0^X$-module L_p such that $Z \otimes_\Gamma \Gamma/(p) \cong F^X(L_p)$.

The admissible seminested tame ditalgebra $\mathcal{A}_0^X$ admits the infinite family $\{L_p\}_{p\in P}$ of pairwise non-isomorphic indecomposable $\mathcal{A}_0^X$-modules with bounded dimension. From [5], (2.10), there are a pregeneric $\mathcal{A}_0^X$-module G', a realization Z' of G', over some rational algebra Γ', and an infinite subset Q of $\mathrm{Irred}(\Gamma')$ such that, for any $q \in Q$, there are $p_q \in P$ and $i_q \in \mathbb{N}$ with

$$Z' \otimes_{\Gamma'} \Gamma'/\big(q^{i_q}\big) \cong L_{p_q} \quad \text{in } \mathcal{A}_0^X\text{-Mod}.$$

Thus, for $q \in Q$, we have

$$Z \otimes_\Gamma \Gamma/(p_q) \cong F^X(L_{p_q}) \cong F^X\big(Z' \otimes_{\Gamma'} \Gamma'/\big(q^{i_q}\big)\big) \cong F^X\big(Z'\big) \otimes_{\Gamma'} \Gamma'/\big(q^{i_q}\big).$$

Moreover, $F^X(G') \cong F^X(Z' \otimes_{\Gamma'} k(x)) \cong F^X(Z') \otimes_{\Gamma'} k(x)$. From 4.18 and [5], (2.5), we obtain that $F^X(G')$ is a pregeneric $\mathcal{A}_0$-module and $F^X(Z')$ is a realization of $F^X(G')$ over Γ'.

Then, from 5.2, we obtain that $F^X(G') \cong G$. Hence, from [5], (3.3), the module $R_0(G)$ is a direct sum of direct summands of X in $\mathcal{A}'_0$-Mod. □

Theorem 5.4. *Assume that $\mathcal{A}'$ is an initial subditalgebra of an almost admissible ditalgebra $\mathcal{A}$, which is constructible from a tame finite-dimensional algebra Λ, over an algebraically closed field k. Consider the extension functor $E = E^{\mathcal{A}}_{\mathcal{A}'} : \mathcal{A}'\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$ and the restriction functor $R = R^{\mathcal{A}}_{\mathcal{A}'} : \mathcal{A}\text{-Mod} \longrightarrow \mathcal{A}'\text{-Mod}$. Then, for any $d \in \mathbb{N}$, there is a finite family $\mathcal{I}(d)$ of finite-dimensional indecomposable $\mathcal{A}'$-modules such that:*

1. *For any indecomposable $\mathcal{A}$-module M with $\dim_k M \leq d$ and $M \ncong E^{\mathcal{A}}_{\mathcal{A}'}(N)$ in $\mathcal{A}$-Mod, for any $N \in \mathcal{A}'$-Mod, the module $R^{\mathcal{A}}_{\mathcal{A}'}(M)$ is isomorphic in $\mathcal{A}'$-Mod to a direct sum of modules in $\mathcal{I}(d)$.*
2. *For any pregeneric $\mathcal{A}$-module G with $\mathrm{endol}(G) \leq d$ and $G \ncong E^{\mathcal{A}}_{\mathcal{A}'}(H)$ in $\mathcal{A}$-Mod, for any pregeneric $\mathcal{A}'$-module H, the module $R^{\mathcal{A}}_{\mathcal{A}'}(G)$ is isomorphic in $\mathcal{A}'$-Mod to a direct sum of modules in $\mathcal{I}(d)$.*

Proof. We first notice that, in order to prove the statement of our theorem, we can assume that $\mathcal{A}$ is admissible. Indeed, if $\mathcal{A}$ is not admissible, we can consider the simultaneous basifications $\mathcal{A}'^b$ of $\mathcal{A}'$ and $\mathcal{A}^b$ of $\mathcal{A}$, as in [4], (3.3). Thus, $\mathcal{A}'^b$ is an initial subditalgebra of the admissible ditalgebra $\mathcal{A}^b$ which is constructible from the finite-dimensional algebra Λ. Moreover, we have the commutative diagram

$$\begin{array}{ccc} \mathcal{A}^b\text{-Mod} & \xrightarrow{F^b} & \mathcal{A}\text{-Mod} \\ \downarrow{\scriptstyle R^b} & & \downarrow{\scriptstyle R} \\ \mathcal{A}'^b\text{-Mod} & \xrightarrow{F'^b} & \mathcal{A}'\text{-Mod} \end{array}$$

where R^b, R denote restriction functors, and F^b, F'^b denote the corresponding equivalence functors. Moreover, $F^bE^b(N) = EF'^b(N)$, for any $N \in \mathcal{A}'^b$-Mod, where $E^b : A'^b\text{-Mod} \longrightarrow A^b\text{-Mod}$ and $E : A'\text{-Mod} \longrightarrow A\text{-Mod}$ are the corresponding extension functors.

Then, if our theorem holds for the ditalgebras $\mathcal{A}'^b$ and $\mathcal{A}^b$, for each $d \in \mathbb{N}$, there is a finite family $\mathcal{I}^b(d)$ of finite-dimensional indecomposable $\mathcal{A}'^b$-modules satisfying the statements corresponding to (1) and (2). Then, it is easy to show that $\mathcal{I}(d) := F'^b(\mathcal{I}^b(d))$ satisfies (1) and (2), using [4], (3.3).

From now on, we assume that $\mathcal{A}$ is admissible.

Then, $\mathcal{A}$ is seminested, by [4], (3.4). Since $\mathcal{A}$ is constructible from a tame algebra, it is pregenerically tame, see [4], (4.6). From [5], (2.8), $\mathcal{A}$ is tame. Now, given a $d \in \mathbb{N}$, we can proceed as in the proof of [3], (4.1), where [6], (28.22) is used to construct a commutative diagram of functors

$$\begin{array}{ccc} \mathcal{A}_0\text{-Mod} & \xrightarrow{F} & \mathcal{A}\text{-Mod} \\ \downarrow{\scriptstyle R_0} & & \downarrow{\scriptstyle R} \\ \mathcal{A}'_0\text{-Mod} & \xrightarrow{F'} & \mathcal{A}'\text{-Mod} \end{array}$$

where: $\mathcal{A}'_0$ is a minimal initial subditalgebra of the seminested ditalgebra $\mathcal{A}_0$; the functors R and R_0 denote the corresponding restrictions; the functors F and F' are compositions of basic reduction functors, and hence are full and faithful, and map finite-dimensional modules onto finite-dimensional modules. Moreover, $FE_0(N'_0) = EF'(N'_0)$, for any $N'_0 \in \mathcal{A}'_0$-Mod, where $E_0 : A'_0\text{-Mod} \longrightarrow A_0\text{-Mod}$ and $E : A'\text{-Mod} \longrightarrow A\text{-Mod}$ are the corresponding extension functors. Furthermore,

(a) we have $\dim_k N_0 \leq \dim_k F(N_0)$, for any $N_0 \in \mathcal{A}_0$-Mod;
(b) for any $M \in \mathcal{A}$-Mod with $\dim_k M \leq d$ there are $N'_0 \in \mathcal{A}'_0$-Mod with $F'(N'_0) \cong R(M)$ and $N_0 \in \mathcal{A}_0$-Mod with $F(N_0) \cong M$ and $R_0(N_0) \cong N'_0$.

Notice that, since $\mathcal{A}'_0$ is a minimal ditalgebra, the canonical embedding functor $L_{\mathcal{A}'_0} : A'_0\text{-Mod} \longrightarrow \mathcal{A}'_0\text{-Mod}$ preserves isomorphism classes. Then, if $M \cong N$ in $\mathcal{A}'_0$-Mod, we have that $M \cong N$ in A'_0-Mod; thus, $E_0(M) \cong E_0(N)$ in A_0-Mod and we get $E_0(M) \cong E_0(N)$ in $\mathcal{A}_0$-Mod. Moreover, the seminested ditalgebra $\mathcal{A}_0$ is biconstructible and, from 5.3, we have a finite family $\mathcal{I}_0(d) = \{X_1, \ldots, X_t\}$ of finite-dimensional indecomposable $\mathcal{A}'_0$-modules satisfying for R_0 and E_0 the statements corresponding to (1) and (2).

Then, we can consider the finite set $\mathcal{I}(d) := F'(\mathcal{I}_0(d))$ consisting of finite-dimensional indecomposable $\mathcal{A}'$-modules. The argument in the last paragraph of the proof of [3], (4.1), using (a) and (b), shows that $\mathcal{I}(d)$ satisfies (1). In order to prove (2), we need:

(a') we have $\mathrm{endol}(N_0) \leq \mathrm{endol}(F(N_0))$, for any $N_0 \in \mathcal{A}_0$-Mod;
(b') for any $M \in \mathcal{A}$-Mod with $\mathrm{endol}(M) \leq d$ there are $N'_0 \in \mathcal{A}'_0$-Mod with $F'(N'_0) \cong R(M)$ and $N_0 \in \mathcal{A}_0$-Mod with $F(N_0) \cong M$ and $R_0(N_0) \cong N'_0$.

The statement (a') follows, for instance, from [4], (2.5)–(2.6) and 4.5(1). Let us prove (b'). Notice that $\mathrm{endol}(R(M)) \leq \mathrm{endol}(M) \leq d$, by [4], (2.2). By construction, for any k-algebra Γ, the functor $F'^{\Gamma} : \mathcal{A}'_0\text{-}\Gamma\text{-Mod} \longrightarrow \mathcal{A}'\text{-}\Gamma\text{-Mod}$ is such that any $\mathcal{A}'$-Γ-bimodule $N' \in \mathcal{A}'\text{-}\Gamma\text{-Mod}$ with $\ell_\Gamma(N') \leq d$ is of the form $N' \cong F'^{\Gamma}(N'_0)$, for some $N'_0 \in \mathcal{A}'_0\text{-}\Gamma\text{-Mod}$, see [6], (28.22). If we make $\Gamma := \mathrm{End}_{\mathcal{A}'}(R(M))^{op}$, then $\ell_\Gamma(R(M)) = \mathrm{endol}(R(M)) \leq d$, and $R(M) \cong F'^{\Gamma}(N'_0)$, for some $N'_0 \in \mathcal{A}'_0\text{-}\Gamma\text{-Mod}$. Then, from [3], (4.5), there is $N_0 \in \mathcal{A}_0\text{-Mod}$ with $F(N_0) \cong M$ and $R_0(N_0) \cong N'_0$.

Now we show that item (2) follows from (a') and (b'). Take a pregeneric $M \in \mathcal{A}\text{-Mod}$ with $\mathrm{endol}(M) \leq d$ and such that $M \ncong E(M')$, for any pregeneric $M' \in \mathcal{A}'\text{-Mod}$. Choose N_0 and N'_0 as in (b'). From (a'), using that N_0 is infinite-dimensional and indecomposable because M is so, we obtain that N_0 is a pregeneric $\mathcal{A}_0$-module with $\mathrm{endol}(N_0) \leq d$. Assume that $N_0 \cong E_0(N''_0)$, for some pregeneric $\mathcal{A}'_0$-module N''_0. Then, $M \cong F(N_0) \cong FE_0(N''_0) = EF'(N''_0)$. Since N''_0 is pregeneric, the module $F'(N''_0)$ is pregeneric too, see for instance [4], (2.5)–(2.6) and 4.7(1). This contradicts our assumption on M. Thus, $\mathrm{endol}(N_0) \leq d$ and $N_0 \ncong E_0(N''_0)$, for any pregeneric $\mathcal{A}'_0$-module N''_0. Then, $R_0(N_0) \cong \bigoplus_i \bigoplus_{J_i} X_i$, for some index sets J_i. Then, $R(M) \cong RF(N_0) = F'R_0(N_0) \cong \bigoplus_i \bigoplus_{J_i} F'(X_i)$ and we are done. □

6. Finite endolength and decompositions

Given a finite-dimensional algebra Λ, many important properties of Λ-modules with finite endolength can be transfered to the category of modules of the constructible ditalgebras from Λ. We discuss some of them in the following.

Proposition 6.1. *Let $\mathcal{A}$ be an almost admissible constructible ditalgebra over the perfect field k and take $M \in \mathcal{A}$-Mod. Then,*

1. *If M has finite endolength, it admits a decomposition of the form $M \cong \bigoplus_{i=1}^t \bigoplus_{I_i} M_i$, for some finite sequence $M_1, \dots, M_t$ of indecomposable $\mathcal{A}$-modules with finite endolength.*
2. *If $M \cong \bigoplus_{i=1}^t \bigoplus_{I_i} M_i$, for some finite sequence $M_1, \dots, M_t$ of indecomposable $\mathcal{A}$-modules with finite endolength, then M has finite endolength.*
3. *If $M \cong \bigoplus_{i \in I} M_i$ in $\mathcal{A}$-Mod and a pregeneric $\mathcal{A}$-module G is a direct summand of M, then G is a direct summand of M_i, for some $i \in I$.*

Proof. Given a finite-dimensional algebra Λ, consider the triangular matrix algebra $\tilde{\Lambda} := \left(\begin{smallmatrix} \Lambda & 0 \\ \Lambda & \Lambda \end{smallmatrix}\right)$. Then, from [6], (18.2), there is an equivalence of categories $\Psi : \tilde{\Lambda}\text{-Mod} \longrightarrow \mathcal{M}(\Lambda)$, where $\mathcal{M}(\Lambda)$ denotes the category of morphisms of Λ-Mod. This equivalence maps each $\tilde{\Lambda}$-module M onto the triple (M_1, M_2, ϕ), where $M_1 = e_1 M$, $M_2 = e_2 M$ and $\phi : M_1 \cong \Lambda \otimes_\Lambda M_1 \longrightarrow M_2$ is determined by the action of $\tilde{\Lambda}$ on M. Here, e_1 and e_2 denote the canonical idempotents of $\tilde{\Lambda}$. From [10], (4.5), each $\tilde{\Lambda}$-module L with finite endolength admits a decomposition of the form $L \cong \bigoplus_{i=1}^t \bigoplus_{I_i} L_i$, for some finite sequence $L_1, \dots, L_t$

of indecomposable $\tilde{\Lambda}$-modules with finite endolength. We can transfer this property to the category $\mathcal{M}(\Lambda)$, with the help of the functor Ψ. Namely, given $Z = (M_1, M_2, \phi) \in \mathcal{M}(\Lambda)$, we can consider the algebra $E_Z := \mathrm{End}_{\mathcal{M}(\Lambda)}(Z)^{op}$, which acts naturally on M_1 and on M_2, hence on $M_1 \oplus M_2$. Let us denote by e_Z the length of $M_1 \oplus M_2$ as an E_Z-module. Notice that, for any $M \in \tilde{\Lambda}$-Mod, we have $\mathrm{endol}(M) = e_{\Psi(M)}$. It is natural to call e_Z the *endolength* of Z, for $Z \in \mathcal{M}(\Lambda)$. Then, we know that any $Z \in \mathcal{M}(\Lambda)$ with finite endolength admits a decomposition of the form $Z \cong \bigoplus_{i=1}^{t} \bigoplus_{I_i} Z_i$, for some $t \in \mathbb{N}$ where each Z_i is an indecomposable of $\mathcal{M}(\Lambda)$ with finite endolength. Now, the category $\mathcal{P}^1(\Lambda)$ is a full subcategory of $\mathcal{M}(\Lambda)$ closed under the formation of direct summands. Thus, $\mathcal{P}^1(\Lambda)$ inherits the preceding property.

Now, consider the usual equivalence $\Xi_\Lambda : \mathcal{D}\text{-Mod} \longrightarrow \mathcal{P}^1(\Lambda)$, where $\mathcal{D}$ is Drozd's ditalgebra of Λ. We claim that, for any $N \in \mathcal{D}$-Mod, the following holds:

$$\mathrm{endol}(N) \leq e_{\Xi_\Lambda(N)} \leq \dim_k \Lambda \times \mathrm{endol}(N).$$

Indeed, make $X := \Xi_\Lambda(N) = (P_1, P_2, \phi)$ in $\mathcal{P}^1(\Lambda)$ and $E := \mathrm{End}_{\mathcal{P}(\Lambda)}(X)^{op}$. Then, we can consider N as a right E-module by restriction, through the isomorphism $E \longrightarrow \mathrm{End}_{\mathcal{D}}(N)^{op}$ given by the quasi inverse of the functor Ξ_Λ. Thus, from [6], (21.10), we have $\mathrm{endol}(N) = \ell_E(N) = \ell_E(P_1/JP_1) + \ell_E(P_2/JP_2) \leq \ell_E(P_1) + \ell_E(P_2) = \ell_E(P_1 \oplus P_2) = e_{\Xi_\Lambda(N)}$, where J denotes the radical of Λ. Moreover, the argument in the proof of [6], (29.5) shows that $\ell_E(P_i) \leq \dim_k \Lambda \times \ell_E(P_i/JP_i)$, for $i \in [1, 2]$. Then, we have $e_{\Xi_\Lambda(N)} = \ell_E(P_1) + \ell_E(P_2) \leq \dim_k \Lambda \times [\ell_E(P_1/JP_1) + \ell_E(P_2/JP_2)] = \dim_k \Lambda \times \mathrm{endol}(N)$, and the claim is proved.

Now, we can transfer the preceding property from $\mathcal{P}^1(\Lambda)$ to the category $\mathcal{D}$-Mod, obtaining the statement (1) of our proposition for the case $\mathcal{A} = \mathcal{D}$.

Finally, consider the almost admissible constructible ditalgebra $\mathcal{A}$ as in 2.3. Then, there is a full and faithful functor $F : \mathcal{A}\text{-Mod} \longrightarrow \mathcal{D}\text{-Mod}$. Take any $M \in \mathcal{A}$-Mod with finite endolength. Then, from [4], (2.5)–(2.6) and 4.7(1), we know that $F(M)$ has finite endolength. It follows that $F(M) \cong \bigoplus_{i=1}^{t} \bigoplus_{I_i} N_i$, where each N_i is an indecomposable $\mathcal{D}$-module with finite endolength. From [6], (29.4), for each N_i there is a direct summand M_i of M with $F(M_i) \cong N_i$. Since F preserves direct sums, we obtain $F(\bigoplus_{i=1}^{t} \bigoplus_{I_i} M_i) \cong \bigoplus_{i=1}^{t} \bigoplus_{I_i} F(M_i) \cong F(M)$. Therefore, $M \cong \bigoplus_{i=1}^{t} \bigoplus_{I_i} M_i$. From [4], (2.5)–(2.6) and 4.6(1), we know that each M_i has finite endolength and we have proved (1).

The proof of (2) follows the same strategy used to prove (1), keeping in mind also [4], (2.7). For (3), we proceed similarly, now using [16], (2.2). □

Lemma 6.2. *Let $\mathcal{A}$ be a layered ditalgebra. Then,*

1. *If M and N are $\mathcal{A}$-modules, we have*

$$\max\left\{\mathrm{endol}(M), \mathrm{endol}(N)\right\} \leq \mathrm{endol}\Big(M \oplus N\Big) \leq \mathrm{endol}(M) + \mathrm{endol}(N).$$

2. If M is an $\mathcal{A}$-module, for any set of indexes I, we have

$$\operatorname{endol}(M) = \operatorname{endol}\left(\bigoplus_I M\right).$$

Proof. The proof given by Crawley-Boevey in [9] for modules over an algebra can be adapted to this case. We give the details.

We can identify $\operatorname{End}_{\mathcal{A}}(M \oplus N)$ with the matrix algebra

$$\begin{pmatrix} \operatorname{End}_{\mathcal{A}}(M) & \operatorname{Hom}_{\mathcal{A}}(N, M) \\ \operatorname{Hom}_{\mathcal{A}}(M, N) & \operatorname{End}_{\mathcal{A}}(N) \end{pmatrix}$$

and consider the canonical morphism of algebras

$$\Gamma := \operatorname{End}_{\mathcal{A}}(M) \times \operatorname{End}_{\mathcal{A}}(N) \xrightarrow{\psi} \operatorname{End}_{\mathcal{A}}(M \oplus N)$$

which maps (f, g) onto the diagonal matrix determined by f and g. Then, by restriction through the morphism ψ, the space $M \oplus N$ has a natural structure of a Γ-module.

(1): For the second inequality in (1), we can assume that $\operatorname{endol}(M) = s$ and $\operatorname{endol}(N) = t$ are finite. Consider a composition series $0 = M_s \subseteq \cdots \subseteq M_1 \subseteq M_0 = M$ of the $\operatorname{End}_{\mathcal{A}}(M)$-module M and a composition series $0 = N_t \subseteq \cdots \subseteq N_1 \subseteq N_0 = N$ of the $\operatorname{End}_{\mathcal{A}}(N)$-module N. Then, the filtration

$$0 = M_s \subseteq \cdots \subseteq M_1 \subseteq M \subseteq M \oplus N_{t-1} \subseteq \cdots \subseteq M \oplus N_1 \subseteq M \oplus N_0 = M \oplus N$$

is a composition series of length $s+t$ of the Γ-module $M \oplus N$ (since each simple Γ-module is either a simple $\operatorname{End}_{\mathcal{A}}(M)$-module or a simple $\operatorname{End}_{\mathcal{A}}(N)$-module). Then, we have that $\ell_\Gamma(M \oplus N) = \operatorname{endol}(M) + \operatorname{endol}(N)$. Moreover, each chain of $\operatorname{End}_{\mathcal{A}}(M \oplus N)$-submodules of $M \oplus N$ is a chain of Γ-submodules of $M \oplus N$, by restriction through ψ. Then, $\operatorname{endol}(M \oplus N) \leq \ell_\Gamma(M \oplus N) = \operatorname{endol}(M) + \operatorname{endol}(N)$.

For the proof of the first inequality in (1), we need the following.

Claim. *For any $\operatorname{End}_{\mathcal{A}}(M)$-submodule X of M, the $\operatorname{End}_{\mathcal{A}}(M \oplus N)$-submodule $\langle X \rangle$ of $M \oplus N$ generated by X satisfies that $X = \langle X \rangle \cap M$.*

Indeed, every element of $\langle X \rangle$ is a finite sum of the form

$$\sum_i \begin{pmatrix} f_i^0 & \zeta_i^0 \\ \theta_i^0 & g_i^0 \end{pmatrix} \begin{pmatrix} x_i \\ 0 \end{pmatrix} = \begin{pmatrix} \sum_i f_i^0(x_i) \\ \sum_i \theta_i^0(x_i) \end{pmatrix},$$

where the square matrices are first components of matrices in the matrix algebra described above and $x_i \in X$. It follows that we have a vector space decomposition $\langle X \rangle = X \oplus Z$, where $Z = \sum_{\theta \in \operatorname{Hom}_{\mathcal{A}}(M,N)} \theta^0(X)$. Hence, $X = \langle X \rangle \cap M$ as claimed.

Now, assume that the $\mathrm{End}_{\mathcal{A}}(M \oplus N)$-module $M \oplus N$ admits a composition series of finite length ℓ. Any chain of $\mathrm{End}_{\mathcal{A}}(M)$-submodules $\cdots \subseteq X_j \subseteq \cdots \subseteq X_1 \subseteq M$ determines the chain $\cdots \subseteq \langle X_j \rangle \subseteq \cdots \subseteq \langle X_1 \rangle \subseteq \langle M \rangle \subseteq M \oplus N$ of $\mathrm{End}_{\mathcal{A}}(M \oplus N)$-submodules of $M \oplus N$. Thus, this last chain must stabilize and refine to a composition series of length ℓ. From *Claim*, the chain $\cdots \subseteq X_j \subseteq \cdots \subseteq X_1 \subseteq M$ stabilizes and refines to a composition series of length $\leq \ell$.

(2): Any composition series $0 = M_n \subseteq \cdots \subseteq M_j \subseteq M_{j-1} \subseteq \cdots \subseteq M_0 = M$ of the $\mathrm{End}_{\mathcal{A}}(M)$-module M determines the composition series $0 = \bigoplus_I M_n \subseteq \cdots \subseteq \bigoplus_I M_j \subseteq \bigoplus_I M_{j-1} \subseteq \cdots \subseteq \bigoplus_I M_0 = \bigoplus_I M$ of the $\mathrm{End}_{\mathcal{A}}(\bigoplus_I M)$-module $\bigoplus_I M$. Thus (2) holds. $\square$

7. Restrictions over real closed fields

Remark 7.1. Denote by K the algebraic closure of the field k and by $[K : k]$ the degree of this extension. Recall that, by Artin–Schreier Theorem, the field k is *real closed* if and only if $1 < [K : k] < \infty$. In this case, k is perfect (in fact, with zero characteristic and $K = k(\sqrt{-1})$, thus $[K : k] = 2$), see [14], §11.7, (11.14) and [14], §11.1, (11.3). For example, the field $\mathbb{R}$ of real numbers is real closed.

Theorem 7.2. *Suppose that $\mathcal{B}$ is an initial subalgebra of an admissible ditalgebra $\mathcal{A}$, over a real closed field k. Assume that $\mathcal{A}$ is constructible from a generically tame finite-dimensional basic algebra Λ. Consider the extension functor $E : \mathcal{B}\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$ and the restriction functor $R : \mathcal{A}\text{-Mod} \longrightarrow \mathcal{B}\text{-Mod}$. Then, for any $d \in \mathbb{N}$, there is a finite family $\mathcal{I}(d)$ of finite-dimensional indecomposable $\mathcal{B}$-modules such that:*

1. *For any indecomposable $\mathcal{A}$-module M with $\dim_k M \leq d$ and $M \ncong E(N)$ in $\mathcal{A}$-Mod, for any $N \in \mathcal{B}$-Mod, the module $R(M)$ is isomorphic in $\mathcal{B}$-Mod to a direct sum of modules in $\mathcal{I}(d)$.*
2. *For any pregeneric $\mathcal{A}$-module M with $\mathrm{endol}(M) \leq d$ and $M \ncong E(N)$ in $\mathcal{A}$-Mod, for any pregeneric $N \in \mathcal{B}$-Mod, the module $R(M)$ is isomorphic in $\mathcal{B}$-Mod to a direct sum of modules in $\mathcal{I}(d)$.*

Proof. Consider the algebraic closure K of the ground field k. Then, we have the commutative squares

$$\begin{array}{ccc} \mathcal{A}\text{-Mod} & \xrightarrow{(-)^K} & \mathcal{A}^K\text{-Mod} \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle \overline{R}} \\ \mathcal{B}\text{-Mod} & \xrightarrow{(-)^K} & \mathcal{B}^K\text{-Mod} \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{A}\text{-Mod} & \xrightarrow{(-)^K} & \mathcal{A}^K\text{-Mod} \\ {\scriptstyle E}\uparrow & & \uparrow{\scriptstyle \overline{E}} \\ \mathcal{B}\text{-Mod} & \xrightarrow{(-)^K} & \mathcal{B}^K\text{-Mod}, \end{array}$$

where E and $\overline{E}$ denote extension functors and R and $\overline{R}$ denote restriction functors, see [6], (20.3) and 4.2. Since k is perfect, by 2.4, the K-ditalgebra $\mathcal{A}^K$ is almost admissible and constructible from Λ^K. From [4], (4.7), it is also pregenerically tame.

Fix $d \in \mathbb{N}$ and make $\bar{d} := 2d$. Then, from 5.4, we know that there is a finite family $\bar{\mathcal{I}}(\bar{d})$ of finite-dimensional indecomposable $\mathcal{B}^K$-modules such that:

1. For any indecomposable $\mathcal{A}^K$-module $\overline{M}$ with $\dim_K \overline{M} \leq \bar{d}$ and such that $\overline{M} \ncong \overline{E}(\overline{N})$, for any $\overline{N} \in \mathcal{B}^K$-Mod, we have that $\overline{R}(\overline{M})$ is isomorphic in $\mathcal{B}^K$-Mod to a direct sum of modules in $\bar{\mathcal{I}}(\bar{d})$.
2. For any pregeneric $\mathcal{A}^K$-module $\overline{M}$ with $\mathrm{endol}(\overline{M}) \leq \bar{d}$ and such that $\overline{M} \ncong \overline{E}(\overline{N})$, for any pregeneric $\overline{N} \in \mathcal{B}^K$-Mod, we have that $\overline{R}(\overline{M})$ is isomorphic in $\mathcal{B}^K$-Mod to a direct sum of modules in $\bar{\mathcal{I}}(\bar{d})$.

Make $\bar{\mathcal{I}}(\bar{d}) = \{\overline{X}_1, \ldots, \overline{X}_m\}$ and consider a finite set of representatives $\mathcal{I}(d)$ of the isomorphism classes of the (necessarily finite-dimensional) indecomposable direct summands of their scalar restrictions $F_\xi(\overline{X}_1), \ldots, F_\xi(\overline{X}_m)$ in $\mathcal{B}$-Mod.

We will see first that $\mathcal{I}(d)$ works for the second item of our theorem. Take $M \in \mathcal{A}$-Mod pregeneric with $\mathrm{endol}(M) \leq d$ and such that $M \ncong E(N)$, for any pregeneric $\mathcal{B}$-module N.

From 3.5, the module M^K has finite endolength $\leq \bar{d}$. Hence, from 6.1, it admits a decomposition $M^K \cong \bigoplus_{i=1}^t \bigoplus_{I_i} \overline{M}_i$ in $\mathcal{A}^K$-Mod as a direct sum of indecomposable $\mathcal{A}^K$-modules $\overline{M}_i$ with finite endolength. From 3.8, none of them is finite-dimensional. Thus, every $\overline{M}_i$ is a pregeneric $\mathcal{A}^K$-module. Moreover, from 6.2, each $\overline{M}_i$ has endolength $\leq \bar{d}$.

Assume that, for some $i \in [1, t]$, we have $\overline{M}_i \cong \overline{E}(\overline{N}_i)$ in $\mathcal{A}^K$-Mod, for some pregeneric $\mathcal{B}^K$-module $\overline{N}_i$. From 3.6 and [4], (2.4), we have

$$\mathrm{endol}\big(F_\xi(\overline{N}_i)\big) \leq \mathrm{endol}(\overline{N}_i) = \mathrm{endol}\big(\overline{E}(\overline{N}_i)\big) = \mathrm{endol}(\overline{M}_i) \leq \bar{d}.$$

Since $\mathcal{B}$-Mod is canonically identified with B-Mod, from [10], (4.5), we have a decomposition $F_\xi(\overline{N}_i) \cong \bigoplus_{j=1}^{s_i} \bigoplus_{J_{ij}} N_{ij}$ as a direct sum of indecomposable $\mathcal{B}$-modules N_{ij} with finite endolength. Again, $\mathcal{B}^K$-Mod is identified with B^K-Mod. Then, from [16], (4.1), the $\mathcal{B}^K$-module $\overline{N}_i$ is a direct summand of $F_\xi(\overline{N}_i)^K \cong \bigoplus_{j=1}^{s_i} \bigoplus_{J_{ij}} N_{ij}^K$. This implies, by [16], (2.2), that the pregeneric module $\overline{N}_i$ is a direct summand of some N_{ij}^K. But, then, $\overline{E}(\overline{N}_i) \cong \overline{M}_i$ is a direct summand of $\overline{E}(N_{ij}^K) \cong E(N_{ij})^K$. From 3.4, we obtain that M is a direct summand of $E(N_{ij})$, which is indecomposable, thus $M \cong E(N_{ij})$. But this contradicts our assumption on M, because N_{ij} is a pregeneric B-module.

Then, $\overline{M}_i \ncong \overline{E}(\overline{N}_i)$, for any pregeneric $\mathcal{B}^K$-module $\overline{N}_i$, for all $i \in [1, t]$. Hence, $\overline{R}(\overline{M}_i)$ is isomorphic in $\mathcal{B}^K$-Mod to a direct sum of modules in $\bar{\mathcal{I}}(\bar{d})$. Hence, the same holds for $\overline{R}(M^K)$. Thus, we obtain that $R(M)^K \cong \overline{R}(M^K) \cong \bigoplus_{r=1}^m \bigoplus_{T_r} \overline{X}_r$ in $\mathcal{B}^K$-Mod, for some index sets $T_1, \ldots, T_m$. Then, applying the scalar restriction functor F_ξ and keeping in mind 3.1, we obtain

$$\coprod_{\mathbb{B}} R(M) \cong F_\xi\big(R(M)^K\big) \cong \bigoplus_{r=1}^m \bigoplus_{T_r} F_\xi(\overline{X}_r).$$

Then, the module $R(M)$ is a direct summand of a direct sum of the finite-dimensional indecomposable B-modules in $\mathcal{I}(d)$. Then, from Warfield's Theorem, the module $R(M)$ is in fact a direct sum of some of these indecomposables, see [1], (26.6). This is what we wanted to prove for item 2.

In order to prove item 1, assume that $M \in \mathcal{A}$-Mod is indecomposable with $\dim_k M \leq d$ and such that $M \ncong E(N)$, for any $N \in B$-Mod. Here again, we can show that $\overline{R}(M^K)$ is a direct sum of modules in $\overline{\mathcal{I}}(\bar{d})$ (the argument is similar to the previous one, see the proof of [4], (5.2)). Then, we proceed as before (where the index sets T_r are now finite of course). □

8. Reduction functors and norms

We shall need the following norm, first introduced in [7], for an induction argument in the proof of Theorem 10.4.

Definition 8.1. Let $\mathcal{A}$ be an admissible k-ditalgebra with layer (R, W) and adopt the notation of 2.2. Fix any k-algebra E. Consider the decomposition $1 = \sum_{i=1}^n e_i$ of the unit of $R = D_1 \times \cdots \times D_n$ as a sum of central primitive orthogonal idempotents. Then, for $M \in \mathcal{A}$-E-Mod, we have its *norm* $\|M\|$ given by

$$\|M\| = \sum_{i,j} \frac{\dim_k(e_i W_0 e_j)}{\dim_k D_i \dim_k D_j} \ell(M_i)\ell(M_j),$$

where ℓ denotes the length of the corresponding right E-module $M_i := e_i M$. The *length vector* of M is $\underline{\ell}(M) = (\ell(M_1), \ldots, \ell(M_n))$. The *support* $\operatorname{supp} M$ of an $\mathcal{A}$-module M is the set of indices $i \in [1, n]$ with $M_i \neq 0$. The $\mathcal{A}$-module is *sincere* iff $\operatorname{supp} M = [1, n]$.

Proposition 8.2. *Let $\mathcal{A}$ be an admissible ditalgebra with layer (R, W), over a perfect field k. Assume that $\mathcal{A}^X$ is the admissible ditalgebra obtained from $\mathcal{A}$ by reduction, using the $\mathcal{B}$-module X, where $\mathcal{B}$ is an initial subalgebra of $\mathcal{A}$ and X is a finite direct sum of pairwise non-isomorphic finite-dimensional indecomposable $\mathcal{B}$-modules. Consider the associated functor $F_X : \mathcal{A}^X$-Mod $\longrightarrow \mathcal{A}$-Mod. Given a fixed k-algebra E, consider the induced functor $F_X^E : \mathcal{A}^X$-E-Mod $\longrightarrow \mathcal{A}$-$E$-Mod. Then, for all $N \in \mathcal{A}^X$-E-Mod, we have that $\|N\| \leq \|F_X^E(N)\|$ and the inequality is strict, whenever $F_X(N)$ is a sincere $\mathcal{A}$-module and $W_0' \neq 0$.*

Proof. By assumption, $\mathcal{A}$ has layer (R, W), where $R = D_1 \times \cdots \times D_n$. Consider the corresponding decomposition $1 = \sum_{i=1}^n e_i$ of the unit of R as a sum of central primitive orthogonal idempotents. By assumption, $\mathcal{B} = (T', \delta')$ is a proper subalgebra associated, say to the R-R-bimodule decomposition $W_0 = W_0' \oplus W_0''$, with $\delta(W_0') = 0$. Make $B = [T']_0 \cong T_R(W_0')$. We also have the admissible finite-dimensional B-module X, with decomposition $X = X_1 \oplus \cdots \oplus X_t$, consisting of pairwise non-isomorphic indecomposable

B-modules. Since k is perfect, there is a splitting $\mathrm{End}_B(X)^{op} = S \oplus P$ over the radical P of the algebra $\mathrm{End}_B(X)^{op}$. Then, the admissible ditalgebra $\mathcal{A}^X$ has layer (S, W^X). Denote by $f_1, \dots, f_t$ the corresponding central primitive orthogonal idempotents of S. Recall that $W_0^X = X^* \otimes_B BW_0''B \otimes_B X \cong X^* \otimes_R W_0'' \otimes_R X$, see [6], (12.2) and [6], (12.7).

For $j \in [1,t]$, write $S_j := Sf_j$ and also, for $N \in \mathcal{A}^X$-Mod, write $N_j := f_jN$. Similarly, write $M_i := e_iM$, for $M \in \mathcal{A}$-Mod and $i \in [1,n]$.

Given $N \in \mathcal{A}^X$-E-Mod, we have $M := F_X^E(N) \in \mathcal{A}$-$E$-Mod, and by 4.4, for $i \in [1,n]$, we obtain the formula

$$\ell(M_i) = \sum_{s=1}^{t} \dim_{S_s}(e_iXf_s)\ell(N_s).$$

Here ℓ denotes the length of the corresponding right E-module M_i or N_s. Notice also that $\dim_{S_s}(e_iXf_s) = \dim_{S_s}(f_sX^*e_i)$, because $f_sX^*e_i = f_s\,\mathrm{Hom}_S(X,S)e_i \cong \mathrm{Hom}_S(e_iX, Sf_s) \cong \mathrm{Hom}_{S_s}(e_iXf_s, S_s)$. Then, we have

$$\begin{aligned}
\|M\| &= \sum_{i,j} \frac{\dim_k(e_iW_0e_j)}{\dim_k D_i \dim_k D_j}\ell(M_i)\ell(M_j) \\
&= \sum_{i,j,s} \frac{\dim_k(e_iW_0e_j)}{\dim_k D_i \dim_k D_j} \dim_{S_s}(e_iXf_s)\ell(N_s)\ell(M_j) \\
&= \sum_{i,j,s} \frac{\dim_k(e_iW_0e_j)}{\dim_k D_i \dim_k D_j} \dim_{S_s}\big(f_sX^*e_i\big)\ell(N_s)\ell(M_j) \\
&= \sum_{i,j,s} \frac{\dim_k(f_sX^*e_i)}{\dim_k S_s} \frac{\dim_k(e_iW_0e_j)}{\dim_k D_i \dim_k D_j}\ell(N_s)\ell(M_j) \\
&= \sum_{i,j,s} \frac{\dim_{D_i}(f_sX^*e_i)\dim_{D_j}(e_iW_0e_j)}{\dim_k S_s}\ell(N_s)\ell(M_j) \\
&= \sum_{j,s} \frac{\dim_{D_j}(f_sX^* \otimes_R W_0e_j)}{\dim_k S_s}\ell(N_s)\ell(M_j) \\
&= \sum_{j,s,r} \frac{\dim_{D_j}(f_sX^* \otimes_R W_0e_j)}{\dim_k S_s}\ell(N_s)\dim_{S_r}(e_jXf_r)\ell(N_r) \\
&= \sum_{s,r} \frac{\dim_{S_r}(f_sX^* \otimes_R W_0 \otimes_R Xf_r)}{\dim_k S_s}\ell(N_s)\ell(N_r) \\
&= \sum_{s,r} \frac{\dim_k(f_sX^* \otimes_R W_0 \otimes_R Xf_r)}{\dim_k S_s \dim_k S_r}\ell(N_s)\ell(N_r) \\
&\geq \sum_{s,r} \frac{\dim_k(f_sX^* \otimes_R W_0'' \otimes_R Xf_r)}{\dim_k S_s \dim_k S_r}\ell(N_s)\ell(N_r) = \|N\|.
\end{aligned}$$

If M is sincere, then we have $M = F_X^E(N) = X \otimes_S N \cong \bigoplus_s Xf_s \otimes_{S_s} f_sN = \bigoplus_{i,s} e_iXf_s \otimes_{S_s} f_sN$. Hence, for any $i \in [1,n]$, there is an $s_i \in [1,t]$ with $e_iXf_{s_i} \otimes_{S_{s_i}} f_{s_i}N \neq 0$. Hence, $f_{s_i}N \neq 0$ and $e_iXf_{s_i} \neq 0$, and also $f_{s_i}X^*e_i \neq 0$. But, since $W_0' \neq 0$, there are $i,j \in [1,n]$ with $e_jW_0'e_i \neq 0$. It follows that $f_{s_j}X^* \otimes_R W_0' \otimes_R Xf_{s_i} \neq 0$. Then, we obtain $\ell(N_{s_i})\ell(N_{s_j}) \neq 0$ and also $[\dim_k S_{s_j} \dim_k S_{s_i}]^{-1} \dim_k(f_{s_j}X^* \otimes_R W_0' \otimes_R Xf_{s_i}) \neq 0$. Hence, $\|N\| < \|M\|$. □

Definition 8.3. Let $\mathcal{A}$ be an admissible ditalgebra, with layer (R,W), and consider the decomposition $1 = \sum_{i=1}^n e_i$ of the unit of R as a sum of central primitive orthogonal idempotents e_i of R. Given $M \in \mathcal{A}\text{-Mod}$, make $E_M := \operatorname{End}_{\mathcal{A}}(M)^{op}$ and define the *endolength vector* of M as

$$\underline{\ell}^e(M) = \big(\ell_1^e(M), \dots, \ell_n^e(M)\big), \quad \text{where } \ell_i^e(M) := \ell_{E_M}(e_iM), \text{ for } i \in [1,n].$$

Thus, $\operatorname{endol}(M) = \sum_{i=1}^n \ell_i^e(M)$. The *endonorm* of M is defined by

$$\|M\|^e := \sum_{i,j} \frac{\dim_k(e_iW_0e_j)}{\dim_k D_i \dim_k D_j} \ell_i^e(M)\ell_j^e(M).$$

For $\underline{\ell} \in \mathbb{Z}^n$, with non-negative entries, its endonorm is defined by

$$\|\underline{\ell}\|^e = \sum_{i,j} \frac{\dim_k(e_iW_0e_j)}{\dim_k D_i \dim_k D_j} \ell_i\ell_j.$$

Remark 8.4. We stress the fact that, in general, the endonorm is not an integral number. For a fixed admissible ditalgebra $\mathcal{A}$, with the preceding notation, we know that for any common multiple c of the set $\{\dim_k D_i \mid i \in [1,n]\}$, the number $c\|M\|^e$ is a non-negative integer, for any $M \in \mathcal{A}\text{-Mod}$ with finite endolength. For the special case of a real closed field k, we get that $4\|M\|^e$ is a non-negative integer, for any admissible ditalgebra $\mathcal{A}$ and any $M \in \mathcal{A}\text{-Mod}$ with finite endolength. This will be of importance later because we shall make inductions related to the endonorm.

The proof of the following lemma is similar to the proof of [4], (7.2).

Lemma 8.5. *Let $\mathcal{A}$ be an admissible ditalgebra with layer (R,W), as in 2.2. Assume that $W_0 \neq 0$ and that M is a sincere indecomposable $\mathcal{A}$-module with $\|M\|^e \leq d$, for some number d. Then,* $\operatorname{endol}(M) \leq ncd$*, for any common multiple c of the set $\{\dim_k D_i \mid i \in [1,n]\}$.*

In particular, if k is a real closed field, we get $\operatorname{endol}(M) \leq 4n\|M\|^e$.

We will need the following result, taken from [7].

Corollary 8.6. *Let $\mathcal{A}$ be an admissible ditalgebra with layer (R,W), over a perfect field k. Let $F^z : \mathcal{A}^z\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$ be the functor associated to the reduction $\mathcal{A} \mapsto \mathcal{A}^z$ of one* *of the types: deletion of idempotents as in [4], (2.5), regularization as in [4], (2.6), or reduction by a $\mathcal{B}$-module as in 8.2. For $N \in \mathcal{A}^z$-Mod, assume that $F^z(N)$ has finite endolength, then we have:*

1. $\|N\|^e = \|F^d(N)\|^e$ *in the deletion of idempotents case;*
2. $\|N\|^e \leq \|F^r(N)\|^e$ *in the regularization case, where the inequality is strict whenever $F^r(N)$ is sincere and $W_0' \neq 0$;*
3. $\|N\|^e \leq \|F^X(N)\|^e$ *in the reduction by a $\mathcal{B}$-module case, where the inequality is strict whenever $F^X(N)$ is sincere and $W_0' \neq 0$.*

Proof. For the third case, we make $E := \mathrm{End}_{\mathcal{A}^X}(N)^{op}$ and recall that $E \cong \mathrm{End}_{\mathcal{A}}(F_X(N))^{op}$. Then, notice that $N \in \mathcal{A}^X$-E-Mod and $F_X^E(N) \in \mathcal{A}$-E-Mod. Moreover, we have $\underline{\ell}^e(N) = \underline{\ell}_E(N)$ and $\underline{\ell}^e(F^X(N)) = \underline{\ell}_E(F_X^E(N))$, then apply 8.2. The first two cases are clear. □

9. Families of modules

Recall from [4] the following definition.

Definition 9.1. A k-algebra B is called *minimal* iff it is of one of the following two types:

1. $B = T_{D_1 \times D_2}(V)$, where D_1 and D_2 are finite-dimensional division k-algebras and V is a simple D_1-D_2-bimodule.
2. $B = T_D(V)$, where D is a finite-dimensional division k-algebra and V is a simple D-D-bimodule.

Remark 9.2. From [4], (6.2), if $B = T_D(V)$ is a pregenerically tame minimal algebra of type 2 in 9.1, then B is a skew polynomial algebra $D[x, s]$, for some automorphism $s : D \longrightarrow D$. It has infinite representation type. Moreover, B admits up to isomorphism, a unique pregeneric module (which is the unique generic module).

As remarked in [4], (6.6), from the work of Dlab, Ringel, and Crawley-Boevey, the description of generically tame minimal algebras of type 1 in 9.1, which are of infinite representation type, is the following. They also admit a unique generic module, up to isomorphism.

1. B is the matrix algebra $\left(\begin{smallmatrix} F & 0 \\ M & G \end{smallmatrix}\right)$, where F and G are finite-dimensional division k-algebras and M is a simple G-F-bimodule where the field k acts centrally. Moreover, $\dim_G M = 2 = \dim M_F$; or
2. B is the matrix algebra $\left(\begin{smallmatrix} F & 0 \\ M & G \end{smallmatrix}\right)$, where F and G are finite-dimensional division k-algebras and M is a simple G-F-bimodule where the field k acts centrally. Moreover, $\dim_G M = 4$ and $\dim M_F = 1$, or $\dim_G M = 1$ and $\dim M_F = 4$.

Remark 9.3. From now on, we assume that the ground field k is real closed. From the Gerstenhaber–Yang Theorem, see [17], (15.10), we know that there are only three isoclasses of finite-dimensional division k-algebras. Namely, the field k itself, its algebraic closure $k(\sqrt{-1})$ and the quaternion algebra over k. For the sake of notational simplicity, and since in this paper we only mention the example of the real numbers field, we abuse of the language and write $k = \mathbb{R}$, denote by $\mathbb{C}$ its algebraic closure and by $\mathbb{H}$ the quaternion algebra. Likewise, we write $i = \sqrt{-1}$ and consider the $\mathbb{R}$-automorphism $\tau : \mathbb{C} \longrightarrow \mathbb{C}$ given by $a + bi \mapsto a - bi$, with $a, b \in \mathbb{R}$, which we call the *conjugation* of $\mathbb{C}$. We also have the *quaternion conjugation* $(-)^* : \mathbb{H} \longrightarrow \mathbb{H}$ given by $(a + bi + cj + dk)^* = a - bi - cj - dk$, which determines an isomorphism of $\mathbb{R}$-algebras $\mathbb{H} \cong \mathbb{H}^{op}$.

Then, as a particular case of the general situation of a hereditary prime noetherian algebra described in [10], (4.7), we have the following (see also [13]).

Lemma 9.4. *If B is a skew polynomial $\mathbb{R}$-algebra, then B is, up to isomorphism, one of the following four $\mathbb{R}$-algebras*

$$\mathbb{R}[x], \quad \mathbb{C}[x], \quad \mathbb{H}[x], \quad \mathbb{C}[x, \tau]$$

where τ is the conjugation automorphism of $\mathbb{C}$. Their corresponding skew fields of fractions $\mathbb{F}(B)$ are, respectively,

$$\mathbb{R}(x), \quad \mathbb{C}(x), \quad \mathbb{H}(x), \quad \mathbb{C}(x, \tau).$$

Then, in each case, the generic B-module G is the ring of fractions $\mathbb{F}(B)$, considered as a B-module by restriction through the embedding $B \longrightarrow \mathbb{F}(B)$, which is an epimorphism of algebras. Thus, the endomorphism algebra of the generic B-module G satisfies that $\mathrm{End}_B(G) \cong \mathbb{F}(B)^{op} \cong \mathbb{F}(B)$.

Lemma 9.5. *Given a real-closed field $\mathbb{R}$, the generically tame finite-dimensional minimal $\mathbb{R}$-algebras of infinite representation type are $\left(\begin{smallmatrix} \mathbb{R} & 0 \\ \mathbb{H} & \mathbb{H} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \mathbb{H} & 0 \\ \mathbb{H} & \mathbb{R} \end{smallmatrix}\right)$.*

Proof. This follows from 9.2 and 9.3. More precisely, from the classification of bimodule $\mathbb{R}$-algebras of tame representation type due to Dlab and Ringel, see the addendum of [11], we know that the representation-infinite tame bimodule $\mathbb{R}$-algebras are isomorphic to one the following six ones:

$$\begin{pmatrix} \mathbb{R} & 0 \\ \mathbb{R} \oplus \mathbb{R} & \mathbb{R} \end{pmatrix} \quad \begin{pmatrix} \mathbb{C} & 0 \\ \mathbb{C} \oplus \mathbb{C} & \mathbb{C} \end{pmatrix} \quad \begin{pmatrix} \mathbb{H} & 0 \\ \mathbb{H} \oplus \mathbb{H} & \mathbb{H} \end{pmatrix} \quad \begin{pmatrix} \mathbb{C} & 0 \\ \mathbb{C} \oplus \mathbb{C}_\tau & \mathbb{C} \end{pmatrix}$$
$$\begin{pmatrix} \mathbb{R} & 0 \\ \mathbb{H} & \mathbb{H} \end{pmatrix} \quad \begin{pmatrix} \mathbb{H} & 0 \\ \mathbb{H} & \mathbb{R} \end{pmatrix}$$

From this list we can discard the first line because the bimodules which appear are not simple. We remain with the last two possibilities. □

The formulation of the following lemma is taken from [13], see also the note added in proof of [12].

Lemma 9.6. *Consider the principal ideal domain* $\mathbb{D} := \mathbb{R}[x,y]/\langle x^2+y^2+1\rangle$ *and its field of fractions* $\mathbb{E} = \mathbb{R}(x)[y]/\langle x^2+y^2+1\rangle$. *As before, the generic* $\mathbb{D}$*-module is* $G = \mathbb{E}$ *seen as a* $\mathbb{D}$*-module via the embedding* $\mathbb{D} \longrightarrow \mathbb{E}$, *and its endomorphism algebra is* $\mathrm{End}_{\mathbb{D}}(G) \cong \mathrm{End}_{\mathbb{E}}(\mathbb{E}) \cong \mathbb{E}$. *For each* $M \in \mathbb{D}\text{-Mod}$, *we can consider the morphism of* $\mathbb{R}$*-algebras* $\nu_M : \mathbb{H} \longrightarrow \mathrm{End}_{\mathbb{R}}(M^2)$ *defined by* $\nu_M(i) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ *and* $\nu_M(j) = \begin{pmatrix} yI & xI \\ xI & -yI \end{pmatrix}$, *which gives* M^2 *a structure of left* $\mathbb{H}$*-module. Then,*

1. *For* $B = \begin{pmatrix} \mathbb{R} & 0 \\ \mathbb{H} & \mathbb{H} \end{pmatrix}$ *there is a full and faithful functor* $H : \mathbb{D}\text{-Mod} \longrightarrow B\text{-Mod}$ *defined by* $H(M) = (M, M^2, \psi_M : \mathbb{H} \otimes_{\mathbb{R}} M \longrightarrow M^2)$, *where* M^2 *is a left* $\mathbb{H}$*-module through* ν_M *and* $\psi_M(h \otimes m) = \nu_M(h)(m,0)^t$. *Then,* $H \cong Y \otimes_{\mathbb{D}} -$, *where* $Y = H(\mathbb{D})$ *is a* B*-*$\mathbb{D}$*-bimodule which is a free* $\mathbb{D}$*-module of rank 3. Thus,* $G := H(\mathbb{E}) \cong Y \otimes_{\mathbb{D}} \mathbb{E}$ *is the generic* B*-module and* $\mathrm{End}_B(G) \cong \mathbb{E}$.
2. *For* $B = \begin{pmatrix} \mathbb{H} & 0 \\ \mathbb{H} & \mathbb{R} \end{pmatrix}$ *there is a full and faithful functor* $H : \mathbb{D}\text{-Mod} \longrightarrow B\text{-Mod}$ *defined by* $H(M) = (M^2, M, \psi_M : \mathbb{H} \otimes_{\mathbb{H}} M^2 \longrightarrow M)$, *where* M^2 *is a left* $\mathbb{H}$*-module through* ν_M *and* $\psi_M(h \otimes (m_1,m_2)^t) = (0,1)\nu_M(h)(m_1,m_2)^t$. *Then,* $H \cong Y \otimes_{\mathbb{D}} -$, *where* $Y = H(\mathbb{D})$ *is a* B*-*$\mathbb{D}$*-bimodule which is a free* $\mathbb{D}$*-module of rank 3. Thus,* $G := H(\mathbb{E}) \cong Y \otimes_{\mathbb{D}} \mathbb{E}$ *is the generic* B*-module and* $\mathrm{End}_B(G) \cong \mathbb{E}$.
3. *In both precedent cases, for each* $d \in \mathbb{N}$ *and almost every indecomposable* $M \in B\text{-Mod}$ *with* $\dim_k M \leq d$, *there is* $N \in \mathbb{D}\text{-Mod}$ *with* $H(N) \cong M$.

In the following statement we resume previous results and fix some notation convenient for later use.

Lemma 9.7. *If* B_i *is some generically tame minimal* $\mathbb{R}$*-algebra of infinite representation type, then:*

1. *If* $B_i \in \{\mathbb{R}[x], \mathbb{C}[x], \mathbb{H}[x], \mathbb{C}[x,\tau]\}$, *write* $\Gamma_i := B_i$ *and denote by* H_i *the tensor product functor* $Y_i \otimes_{\Gamma_i} - : \Gamma_i\text{-Mod} \longrightarrow B_i\text{-Mod}$, *where* $Y_i = \Gamma_i$.
2. *If* $B_i = \begin{pmatrix} \mathbb{R} & 0 \\ \mathbb{H} & \mathbb{H} \end{pmatrix}$ *write* $\Gamma_i := \mathbb{D}$, *denote respectively by* Y_i *and* H_i *the bimodule* Y *and the functor* H *described in 9.6(1).*
3. *If* $B_i = \begin{pmatrix} \mathbb{H} & 0 \\ \mathbb{H} & \mathbb{R} \end{pmatrix}$ *write* $\Gamma_i := \mathbb{D}$, *denote respectively by* Y_i *and* H_i *the bimodule* Y *and the functor* H *described in 9.6(2).*

Then, in each case, $H_i : \Gamma_i\text{-Mod} \longrightarrow B_i\text{-Mod}$ *is a full and faithful tensor product functor defined by a* B_i*-*Γ_i*-bimodule* Y_i, *which is free of finite rank as a right* Γ_i*-module. It maps the skew field of fractions* Q_i *of* Γ_i *onto the generic* B_i*-module. Moreover,* Y_i *is a realization of* $H_i(Q_i)$ *over* Γ_i, *see 4.19. For each* $d \in \mathbb{N}$ *and almost every indecomposable* $M \in B_i\text{-Mod}$ *with* $\dim_k M \leq d$, *there is* $N \in \Gamma_i\text{-mod}$ *with* $H_i(N) \cong M$. *We use* $\Gamma_i\text{-mod}$ *to denote the category of finite-length* Γ_i*-modules.*

We consider the family $\mathcal{R} = \{\mathbb{R}[x], \mathbb{C}[x], \mathbb{H}[x], \mathbb{C}[x, \tau], \mathbb{D}\}$, which consists of (non-necessarily commutative) principal ideal domains.

Let us reformulate in the following theorem some results of [4], for the real closed field case.

Theorem 9.8. *Assume that an admissible ditalgebra $\mathcal{A}$ is constructible from a generically tame finite-dimensional basic algebra over a real closed field k. Fix $d \in \mathbb{N}$. Then, there are constructible ditalgebras $\mathcal{A}_1, \dots, \mathcal{A}_m$, generically tame minimal algebras of infinite representation type $B_1, \dots, B_m$, where each B_i is an initial subalgebra of $\mathcal{A}_i$, principal ideal domains $\Gamma_1, \dots, \Gamma_m \in \mathcal{R}$, and a family of functors $\hat{F}_1, \dots, \hat{F}_m$ satisfying the following:*

1. *Each functor $\hat{F}_i : \Gamma_i\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$ preserves indecomposability and isomorphism classes, for any $i \in [1, m]$;*
2. *For almost every indecomposable module $M \in \mathcal{A}\text{-Mod}$ with $\dim_k M \leq d$ there are $i \in [1, m]$ and $N \in \Gamma_i\text{-mod}$ such that $\hat{F}_i(N) \cong M$ in $\mathcal{A}\text{-Mod}$.*
3. *If $\{N_u\}_{u \in U}$ and $\{M_u\}_{u \in U}$ are infinite families of pairwise non-isomorphic indecomposable modules in $\Gamma_i\text{-mod}$ and $\Gamma_j\text{-mod}$, respectively, such that $\hat{F}_i(N_u) \cong \hat{F}_j(M_u)$, for all $u \in U$, then $i = j$.*
4. *Each functor $\hat{F}_i$ is given by the following composition*

$$\Gamma_i\text{-Mod} \xrightarrow{H_i} B_i\text{-Mod} \xrightarrow{E_i} \mathcal{A}_i\text{-Mod} \xrightarrow{F_i} \mathcal{A}\text{-Mod},$$

where E_i is the associated extension functor, F_i is the composition of reduction functors associated to a finite sequence of reductions which transform $\mathcal{A}$ into $\mathcal{A}_i$, the algebra Γ_i associated to B_i and the functor $H_i = Y_i \otimes_{\Gamma_i} - : \Gamma_i\text{-Mod} \longrightarrow B_i\text{-Mod}$ are those described in 9.7.

Proof. It follows from [4], (7.6) and 9.7, see [4], (7.3)(4), [4], (7.7), and [4], (10.1). □

Remark 9.9. If $\mathcal{B}$ is a proper subalgebra of the Roiter ditalgebra $\mathcal{A}$, $E : \mathcal{B}\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$ is the extension functor, and $M \in \mathcal{B}\text{-Mod}$, then

$$\operatorname{End}_{\mathcal{A}}\big(E(M)\big)/\operatorname{rad}\operatorname{End}_{\mathcal{A}}\big(E(M)\big) \cong \operatorname{End}_{\mathcal{B}}(M)/\operatorname{rad}\operatorname{End}_{\mathcal{B}}(M).$$

Indeed, the restriction functor $R : \mathcal{A}\text{-Mod} \longrightarrow \mathcal{B}\text{-Mod}$ determines a surjective morphism of algebras $\operatorname{End}_{\mathcal{A}}(E(M)) \longrightarrow \operatorname{End}_{\mathcal{B}}(M)$ which induces, by [6], (31.6) and [6], (5.8), the required isomorphism.

Theorem 9.10. *Assume that an admissible ditalgebra $\mathcal{A}$ is constructible from a generically tame finite-dimensional basic algebra over a real closed field k. Then, for each $d \in \mathbb{N}$, there are pregeneric $\mathcal{A}$-modules $G_1, \dots, G_m$ and, for each $i \in [1, m]$, a realization Z_i of G_i over an algebra $\Gamma_i \in \mathcal{R}$, which is free as a right Γ_i-module, such that*

1. *For any* $i \in [1, m]$*, the composition* $\Gamma_i\text{-Mod} \xrightarrow{Z_i \otimes_{\Gamma_i} -} \mathcal{A}\text{-Mod} \xrightarrow{L_{\mathcal{A}}} \mathcal{A}\text{-Mod}$ *preserves indecomposables and isomorphism classes.*
2. *For almost all indecomposable* $\mathcal{A}$*-module* M *with* $\dim_k M \leq d$*, we have an isomorphism* $M \cong Z_i \otimes_{\Gamma_i} N$ *in* $\mathcal{A}$-Mod*, for some* $i \in [1, m]$ *and some indecomposable* $N \in \Gamma_i$-mod*.*
3. *Given* $i \in [1, m]$*, if* Q_i *denotes the skew field of fractions of* Γ_i*, then*

$$\operatorname{End}_{\mathcal{A}}(G_i)/\operatorname{rad}\operatorname{End}_{\mathcal{A}}(G_i) \cong Q_i.$$

Proof. Apply 9.8 to a given d and adopt the notation given there. For each $i \in [1, m]$, consider the generic Γ_i-module Q_i and its image $G_i := \hat{F}_i(Q_i)$ under the functor $\hat{F}_i$, which is the following composition

$$\Gamma_i\text{-Mod} \xrightarrow{H_i} B_i\text{-Mod} \xrightarrow{E_i} \mathcal{A}_i\text{-Mod} \xrightarrow{F_i} \mathcal{A}\text{- Mod}.$$

Then, by 9.7 and [4], (2.4)–(2.7), the $\mathcal{A}$-module G_i is pregeneric. From [6], (22.7), the following diagram commutes up to isomorphism

$$\begin{array}{ccccc}
\Gamma_i\text{-Mod} & \xrightarrow{E_i(Y_i)\otimes_{\Gamma_i} -} & A_i\text{-Mod} & \xrightarrow{L_{\mathcal{A}_i}} & \mathcal{A}_i\text{-Mod} \\
\| & & \downarrow{\scriptstyle \underline{F}_i} & & \downarrow{\scriptstyle F_i} \\
\Gamma_i\text{-Mod} & \xrightarrow{F_iE_i(Y_i)\otimes_{\Gamma_i} -} & A\text{-Mod} & \xrightarrow{L_{\mathcal{A}}} & \mathcal{A}\text{-Mod}.
\end{array}$$

The composition $\hat{F}_i \cong F_i L_{\mathcal{A}_i}[E_i(Y_i) \otimes_{\Gamma_i} -]$ preserves indecomposability and isomorphism classes. This implies the same property for the composition functor $L_{\mathcal{A}}(F_iE_i(Y_i) \otimes_{\Gamma_i} -)$ in the lower row of the diagram.

Since F_i is a reduction functor, the A-Γ_i-bimodule $Z_i := F_iE_i(Y_i)$ is projective and finitely generated by the right. But Γ_i is a principal ideal domain, thus Z_i is in fact a free right Γ_i-module of finite rank.

We have in $\mathcal{A}$-Mod the isomorphisms $Z_i \otimes_{\Gamma_i} Q_i = L_{\mathcal{A}}(F_iE_i(Y_i) \otimes_{\Gamma_i} Q_i) \cong F_i L_{\mathcal{A}_i}(E_i(Y_i) \otimes_{\Gamma_i} Q_i) \cong F_iE_iH_i(Q_i) \cong \hat{F}_i(Q_i) = G_i$. From [6], (31.6) and 9.9, keeping in mind that F_i and H_i are full and faithful, we have

$$\begin{aligned}
\operatorname{End}_{\mathcal{A}}(G_i)/\operatorname{rad}\operatorname{End}_{\mathcal{A}}(G_i) &\cong \operatorname{End}_{\mathcal{A}}\bigl(F_iE_iH_i(Q_i)\bigr)/\operatorname{rad}\operatorname{End}_{\mathcal{A}}\bigl(F_iE_iH_i(Q_i)\bigr) \\
&\cong \operatorname{End}_{\Gamma_i}(Q_i)/\operatorname{rad}\operatorname{End}_{\Gamma_i}(Q_i) \cong Q_i^{op} \cong Q_i.
\end{aligned}$$

Since the functors E_i and F_i preserve realizations, from 9.7, we obtain the wanted realization Z_i for G_i over the algebra Γ_i, and we have proved (1) and (3).

Moreover, from 9.8, we know that for almost every indecomposable $\mathcal{A}$-module M with $\dim_k M \leq d$, we have an isomorphism $M \cong \hat{F}_i(N) \cong Z_i \otimes_{\Gamma_i} N$ in $\mathcal{A}$-mod for some $i \in [1, m]$ and some $N \in \Gamma_i$-mod. $\square$

10. Reduction to minimal algebras

Remark 10.1. Assume that $\mathcal{A}$ is an admissible ditalgebra over a perfect field as in 2.2 and consider the decomposition $1 = \sum_{i=1}^{n} e_i$ of the unit of the algebra R as a sum of central primitive orthogonal idempotents. When dealing with the transformation of length vectors of modules under reduction functors, it is convenient to introduce the following matrices. Assume that the ditalgebra $\mathcal{A}^z$ is obtained from $\mathcal{A}$ by some reduction of type [4], (2.5), [4], (2.6) or 8.2, and denote by $F^z : \mathcal{A}^z\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$ the corresponding reduction functor (for $z \in \{d, r, X\}$). Then, we can consider the following matrices with non-negative integral entries:

1. In the regularization case, we have the matrix $[F^r]$ which is just the identity $n \times n$ matrix;
2. In the deletion of idempotents case, say that we delete the idempotents $\{e_{t+1}, \dots, e_n\}$, for some $t \in [1, n-1]$, we have the $(n \times t)$-matrix $[F^d]$ such that $[F^d]_{i,i} = 1$ if $i \in [1, t]$, and all the other components are zero;
3. In the reduction of a $\mathcal{B}$-module case, we are assuming that $\mathcal{B}$ is an initial subalgebra of $\mathcal{A}$ and that X is a direct sum of pairwise non-isomorphic finite-dimensional indecomposable $\mathcal{B}$-modules. Consider a splitting of the radical $\mathrm{End}_{\mathcal{B}}(X)^{op} = S \oplus P$ and the canonical decomposition $1 = \sum_{j=1}^{t} f_j$ of the unit of S as a sum of central primitive orthogonal idempotents. Then, there are two interesting $(n \times t)$-matrices to consider: the matrix $[F^X]^e$ defined by $[F^X]^e_{i,j} := \dim_{Sf_j}(e_i X f_j)$, as considered in 4.4, and the matrix $[F^X]$ defined by $[F^X]_{i,j} := \dim_{Re_i}(e_i X f_j)$, as considered in [4], (7.3). These matrices are related by the formula

$$[F^X]^e_{i,j} = \frac{[Re_i : k]}{[Sf_j : k]}[F^X]_{i,j}.$$

Let us write $[F^r]^e := [F^r]$ and $[F^d]^e := [F^d]$, and also write $\underline{\ell}(M)$ for the length vector of the $\mathcal{A}$-module M over the underlying algebra of the layer of $\mathcal{A}$. Then, for any z and $M \in \mathcal{A}^z\text{-Mod}$, we have the formulas

$$\begin{cases} \underline{\ell}(F^z(M)) = [F^z]\underline{\ell}(M)^t \\ \underline{\ell}^e(F^z(M)) = [F^z]^e\underline{\ell}^e(M)^t, \end{cases}$$

which follow from [4], (7.3) and 4.4, taking $E = \mathrm{End}_{\mathcal{A}^X}(M)^{op} \cong \mathrm{End}_{\mathcal{A}}(F^X(M))^{op}$ in case $z = X$.

If $F : \mathcal{A}'\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$ is a composition of reduction functors, say $F = F^{z_s} \cdots F^{z_1}$, with $z_i \in \{d, r, X\}$, then its action on length vectors and endolength vectors is determined, respectively, by the matrices $[F] := [F^{z_s}] \cdots [F^{z_1}]$ and $[F]^e := [F^{z_s}]^e \cdots [F^{z_1}]^e$.

An interesting fact is that, assuming that the underlying algebra of the layer of $\mathcal{A}'$ has u indecomposable factors, we have:

$$\mathrm{supp}\big([F]\underline{\ell}^t\big) = \mathrm{supp}\big([F]^e\underline{\ell}^t\big), \quad \text{for any } \underline{\ell} \in \mathbb{Z}^u \text{ with non-negative entries.}$$

Here, $\mathrm{supp}(\underline{\ell})$ denotes the set of indexes $i \in [1,u]$ with $\ell_i \neq 0$. In particular, given any $\mathcal{A}'$-modules M and N such that $\underline{\ell}(M)$ and $\underline{\ell}^e(N)$ are linearly dependent, then the module $F(M)$ is sincere iff so is the module $F(N)$.

Lemma 10.2. *Assume that k is a real closed field and that B is a generically tame finite-dimensional minimal algebra of infinite representation type. Then the endolength vector of the unique generic B-module belongs to the radical of the quadratic form of B.*

Proof. In the first case listed in 9.5, the generic module has the form described in 9.6(1). Hence, its endolength vector $(1,2)$ generates the radical of the quadratic form of B. The other case is similar. □

Lemma 10.3. *Let $\mathcal{A}$ be an admissible ditalgebra over any field k, and adopt the notation of 2.2. Assume that G is a pregeneric $\mathcal{A}$-module with a realization Z over some principal ideal k-domain Γ such that Z is free as a right Γ-module. Assume that a non-zero module $M \in \mathcal{A}\text{-mod}$ has the form $M \cong Z \otimes_\Gamma N$, for some $N \in \Gamma\text{-mod}$. Then,*

$$\|M\| = (\dim_k N)^2 \|G\|^e,$$

where $\|M\| = \sum_{i,j} \dim_{D_i}(e_iM)\dim_{D_j}(e_jM)\dim_k(e_iW_0e_j)$ is the norm used systematically in [4]. Moreover, the modules M and G share the same support. In particular, M is sincere if and only if G is sincere.

Proof. We know that $G \cong Z \otimes_\Gamma Q$, where Q is the skew field of fractions of Γ, and $\mathrm{endol}(G) = \dim_Q(Z\otimes_\Gamma Q)$. By assumption Z is a free right Γ-module, then the Γ-module e_iZ is projective and in fact free (say of rank r_i) as a right Γ-module. Then, $\underline{\ell}^e(G) = \underline{\ell}^e(Z \otimes_\Gamma Q) = (r_1,\dots,r_n)$. Indeed, since Q embeds in $E_{Z\otimes_\Gamma Q}$ through $q \mapsto (1 \otimes \mu_q, 0)$, where μ_q is right multiplication by q, any composition series of the $E_{Z\otimes_\Gamma Q}$-module $e_iZ \otimes_\Gamma Q$ is a filtration of Q-vector spaces, thus $\ell_{E_{Z\otimes_\Gamma Q}}(e_iZ \otimes_\Gamma Q) \leq \dim_Q(e_iZ \otimes_\Gamma Q)$. Then,

$$\mathrm{endol}(Z \otimes_\Gamma Q) = \sum_i \ell_{E_{Z\otimes_\Gamma Q}}(e_iZ \otimes_\Gamma Q) \leq \sum_i \dim_Q(e_iZ \otimes_\Gamma Q) = \dim_Q(Z \otimes_\Gamma Q),$$

and we obtain the equalities $\ell_{E_{Z\otimes_\Gamma Q}}(e_iZ \otimes_\Gamma Q) = \dim_Q(e_iZ \otimes_\Gamma Q) = r_i$, as claimed. Moreover, $\dim_k(e_iM) = \dim_{D_i}(e_iM)\dim_k D_i$. Also, $M \cong Z \otimes_\Gamma N$, implies $e_iM \cong e_iZ \otimes_\Gamma N \cong r_iN$. Then, we obtain that $\underline{\ell}^e(G)\dim_k N = (\dim_k e_1M,\dots,\dim_k e_nM)$. From here our lemma follows. □

For the statement of the following theorem and the next one, we agree that, given a generically tame infinite-dimensional minimal algebra B of infinite representation type, to call *regular* any finite-dimensional B-module. For the remaining generically tame minimal algebras B of infinite representation type, those considered in 9.5, it already makes sense to consider the regular B-modules, see [4], (6.6).

Theorem 10.4. *Assume that an admissible ditalgebra $\mathcal{A}$ is constructible from a generically tame finite-dimensional basic algebra over a real closed field k. Then, for any integer $d \geq 0$, there are constructible ditalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_m$, generically tame minimal algebras of infinite representation type $B_1, \ldots, B_m$, where each B_i is an initial subalgebra of $\mathcal{A}_i$, and a family of functors $F_1, \ldots, F_m$ such that:*

1. *The functor $F_i : B_i\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$ preserves indecomposability and isomorphism classes, for any $i \in [1, m]$.*
2. *For each sincere pregeneric $G \in \mathcal{A}\text{-Mod}$ with $\|G\|^e \leq d$ there is a unique $i \in [1, m]$ such that $F_i(H_i) \cong G$ in $\mathcal{A}\text{-Mod}$, where H_i denotes the unique generic B_i-module.*
3. *For almost every sincere indecomposable $M \in \mathcal{A}\text{-mod}$ with $\|M\| \leq d$ there are $i \in [1, m]$ and $N \in B_i\text{-mod}$ such that $F_i(N) \cong M$ in $\mathcal{A}\text{-Mod}$.*
4. *The functor $F_i : B_i\text{-mod} \longrightarrow \mathcal{A}\text{-mod}$ maps regular B_i-modules onto sincere $\mathcal{A}$-modules, for each $i \in [1, m]$.*
5. *If $\{N_u\}_{u \in U}$ and $\{M_u\}_{u \in U}$ are infinite families of pairwise non-isomorphic indecomposable regular modules in B_i-mod and B_j-mod, respectively, such that $F_i(N_u) \cong F_j(M_u)$ for all $u \in U$, then $i = j$.*
6. *Each functor F_i is the composition $B_i\text{-Mod} \xrightarrow{E_i} \mathcal{A}_i\text{-Mod} \xrightarrow{G_i} \mathcal{A}\text{-Mod}$, where E_i is the associated extension functor and G_i is the composition of the reduction functors associated to a finite sequence of reductions which transform $\mathcal{A}$ into $\mathcal{A}_i$.*

Proof. This proof is an adaptation of the proof of [4], (7.5), but here we deal with pregeneric modules in addition to finite-dimensional indecomposables, and we use simultaneously the endonorm and the usual norm for the induction. So we give a full proof.

Notice first that the statement of our Theorem 10.4 follows from the statement 10.4', obtained from 10.4 replacing item *2* by the following:

2'. *For each sincere pregeneric $G \in \mathcal{A}\text{-Mod}$ with $4\|G\|^e \leq d$ there is a unique $i \in [1, m]$ such that $F_i(H_i) \cong G$ in $\mathcal{A}\text{-Mod}$, where H_i denotes the unique generic B_i-module.*

Indeed, given $d \geq 0$, we can apply 10.4' to the integer $d' = 4d$ to obtain 10.4 for d. Of course, we deal with $4\|G\|^e$ instead of $\|G\|^e$ in order to guarantee that $4\|G\|^e < d$ implies that $4\|G\|^e \leq d - 1$.

Since $\mathcal{A}$ is constructible, from [4], (4.6), we know that $\mathcal{A}$ is pregenerically tame. The same will remain true for any ditalgebra obtained from $\mathcal{A}$ by a finite number of reductions.

We shall prove 10.4' by induction on d. In this proof we shall say that an admissible ditalgebra $\mathcal{A}$ is (resp. *sincerely*) *d-trivial* iff there is only a finite number of isoclasses of (resp. sincere) indecomposable $\mathcal{A}$-modules M with $\|M\| \leq d$, and there is no (resp. sincere) pregeneric $\mathcal{A}$-module G with $4\|G\|^e \leq d$.

If $d = 0$, then $\mathcal{A}$ is sincerely d-trivial. Indeed, if $G \in \mathcal{A}$-Mod is sincere pregeneric with $\|G\|^e = 0$ then, the layer of $\mathcal{A}$ is of the form (R, W), with $W_0 = 0$. Hence, $\mathcal{A}$ admits no pregeneric modules (in fact, there is no infinite-dimensional indecomposable $\mathcal{A}$-module); similarly, if $M \in \mathcal{A}$-Mod is a sincere indecomposable $\mathcal{A}$-module with $\|M\| = 0$, again the layer of $\mathcal{A}$ is of the form (R, W), with $W_0 = 0$ and there is only a finite number of isoclasses of finite-dimensional indecomposables.

So if $d = 0$, there is nothing to show: the empty family of functors works. So assume that $d > 0$ and that 10.4' holds for any admissible ditalgebra $\mathcal{A}'$ constructible from a generically tame finite-dimensional basic algebra and any $d' < d$.

Now, we have to consider the sincere pregeneric modules $G \in \mathcal{A}$-Mod with $4\|G\|^e \leq d$ (for item *2'*) and the sincere indecomposable modules $M \in \mathcal{A}$-mod with $\|M\| \leq d$ (for item *3*). Again, if $\mathcal{A}$ admits no such modules G, and only finitely many isoclasses of such modules M, we have nothing to show (the empty family of functors works). So we assume that $\mathcal{A}$ is not sincerely d-trivial.

Since $\mathcal{A}$ is an admissible ditalgebra, we can look at the triangular filtration $0 = W_0^0 \subseteq W_0^1 \subseteq \cdots \subseteq W_0^s = W_0$, which is additive because the field k is perfect and hence $R \otimes_k R$ is semisimple. Then, after performing a refinement, if necessary, we can assume that W_0^1 is a simple direct summand of the R-R-bimodule W_0. Then, by triangularity, we have that $\delta(W_0^1) \subseteq W_1$. Moreover, we have R-R-bimodule decompositions $W_0 = W_0^1 \oplus W_0''$ and $W_1 = \delta(W_0^1) \oplus W_1''$. We consider two cases.

Case 1. $\delta(W_0^1) \neq 0$.

Since W_0^1 is a simple R-R-bimodule, $W_0^1 \cap \operatorname{Ker} \delta = 0$ and we can apply the regularization procedure described in [4], (2.6), to obtain an equivalence functor $F^r : \mathcal{A}^r\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$, as in [6], (8.19). If $\mathcal{A}^r$ is sincerely $(d-1)$-trivial, then $\mathcal{A}$ is sincerely d-trivial. Indeed, given any sincere pregeneric $\mathcal{A}$-module G with $4\|G\|^e \leq d$, there is $H \in \mathcal{A}^r$-Mod with $F^r(H) \cong G$ and $\|H\|^e < \|G\|^e$, and so H is a sincere pregeneric $\mathcal{A}^r$-module with $4\|H\|^e \leq d - 1$. Similarly, an infinite family $\{M_t\}_t$ of pairwise non-isomorphic sincere indecomposable $\mathcal{A}$-modules with $\|M_t\| \leq d$ gives rise to a family $\{N_t\}_t$ of pairwise non-isomorphic sincere indecomposable $\mathcal{A}^r$-modules with $F^r(N_t) \cong M_t$ and $\|N_t\| \leq d - 1$.

By assumption, $\mathcal{A}$ is not sincerely d-trivial, thus $\mathcal{A}^r$ is not sincerely $(d-1)$-trivial and we can apply the induction hypothesis to the constructible pregenerically tame ditalgebra $\mathcal{A}^r$ and $d - 1$, to obtain a family of functors $F_i : B_i\text{-Mod} \longrightarrow \mathcal{A}^r\text{-Mod}$, $i \in [1, m]$, satisfying the corresponding conditions *1, 2', 3–6*. Let us show that $\mathcal{F} := \{F^r F_i \mid i \in [1, m]\}$ is the required family of functors for $\mathcal{A}$ and d.

Item *1* is clear because F^r preserves indecomposables and isomorphism classes. Hence, so does every functor in $\mathcal{F}$. Item *2'* is also clear, since we realize every sincere pregeneric $\mathcal{A}$-module G with $4\|G\|^e \leq d$ as $F^r(N) \cong G$, for some $\mathcal{A}^r$-module N with $4\|N\|^e < d$. Then, we can apply our induction hypothesis to such sincere pregeneric $\mathcal{A}^r$-module N to obtain $F_i(H) \cong N$, for some generic $H \in B_i$-Mod, thus $G \cong F^r F_i(H)$ and we are done.

Item *3* is also clear, since we realize every sincere indecomposable $\mathcal{A}$-module M with $\|M\| \leq d$ as $F^r(N) \cong M$, for some $\mathcal{A}^r$-module N with $\|N\| < d$. Then, we can apply our induction hypothesis to such sincere indecomposable $\mathcal{A}^r$-module N to obtain $F_i(H) \cong N$, for some $H \in B_i$-mod, thus $M \cong F^r F_i(H)$ and we are done.

Item *4* follows from the induction hypothesis and the fact that the functor F^r maps sincere modules onto sincere modules. Item *5* follows from the induction hypothesis and the fact that F^r reflects isomorphisms. Item *6* is clear.

Case 2. $\delta(W_0^1) = 0$.

Consider the initial subalgebra $\mathcal{B}$ of $\mathcal{A}$ determined by the R-R-bimodule W_0^1 given above. Consider also the extension functor $E : \mathcal{B}\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$ and the restriction functor $R : \mathcal{A}\text{-Mod} \longrightarrow \mathcal{B}\text{-Mod}$. Let us examine the algebra B.

Since W_0^1 is a simple R-R-bimodule, there exist $i, j \in [1, n]$ with $e_j W_0^1 e_i = W_0^1$. If $i = j$, for notational simplicity, we assume that $i = 1$ and write $e := e_1$. Then, $B = T_R(W_0^1) \cong T_{D_1}(W_0^1) \times D_2 \times \cdots \times D_n$. Thus $eBe = T_{D_1}(W_0^1)$ is a minimal algebra. If $i \neq j$, for notational simplicity, we assume that $i = 1$, $j = 2$ and write $e = e_1 + e_2$. Then, $B = T_R(W_0^1) \cong T_{D_1 \times D_2}(W_0^1) \times D_3 \times \cdots \times D_n$. Then, $eBe = T_{D_1 \times D_2}(W_0^1)$ is a minimal algebra.

In both cases, the extension functor $Be \otimes_{eBe} - : eBe\text{-Mod} \longrightarrow B\text{-Mod}$ is full, faithful, and such that, with only a finite number of possible exceptions, the isoclasses of the indecomposable B-modules are represented by modules of the form $Be \otimes_{eBe} H$, for some indecomposable $H \in eBe$-Mod. The exceptions are some finite-dimensional B-modules.

Since $\mathcal{A}$ is pregenerically tame, from [4], (2.4), we know that B is also pregenerically tame. It follows that eBe is pregenerically tame too. We can consider the composition functor F

$$eBe\text{-Mod} \xrightarrow{Be \otimes_{eBe} -} B\text{-Mod} \xrightarrow{E} \mathcal{A}\text{-Mod},$$

which, from [4], (2.4), preserves indecomposability and isomorphism classes. Notice that if F maps an indecomposable eBe-module onto a sincere $\mathcal{A}$-module, then $B = eBe$. We will need the following claim.

Claim. *Assume that eBe has infinite representation type. Then, if F maps any indecomposable eBe-module onto a sincere $\mathcal{A}$-module, it maps each indecomposable regular eBe-module onto a sincere $\mathcal{A}$-module.*

Proof of Claim. Assume that F maps an indecomposable eBe-module onto a sincere $\mathcal{A}$-module. Thus, $eBe = B$. If $i = j$ and $0 \neq H \in B$-Mod, then $F(H)$ not sincere means that the number n of idempotents in R is > 1, and hence $eBe \neq B$, a contradiction. Then, $i \neq j$. As remarked in [4], (6.6), the generator $\underline{\lambda}$ of the radical of the quadratic form of eBe is sincere and such that, for any indecomposable regular B-module H, we have $\underline{\ell}(H) = c_H \underline{\lambda}$, for some $c_H \in \mathbb{N}$. Then $\underline{\ell}(F(H)) = c_H \underline{\lambda}$ has no zero components, and $F(H)$ is sincere. □

In case the algebra B is not of finite representation type, apply 7.2 to the number $d' := 4nd$, to obtain a finite family $\mathcal{I}(d')$ of finite-dimensional indecomposable B-modules such that:

- For any pregeneric $\mathcal{A}$-module G with $\mathrm{endol}(G) \leq d'$ and $G \not\cong E(N)$ in $\mathcal{A}$-Mod, for any pregeneric $N \in B$-Mod, the module $R(G)$ is isomorphic in B-Mod to a direct sum of modules in $\mathcal{I}(d')$.
- For any indecomposable $\mathcal{A}$-module M with $\dim_k M \leq d'$ and $M \not\cong E(N)$ in $\mathcal{A}$-Mod, for any $N \in B$-Mod, the module $R(M)$ is isomorphic in B-Mod to a direct sum of modules in $\mathcal{I}(d')$.

If we are in the case where the algebra B is of finite representation type, we denote by $\mathcal{I}(d')$ a complete set of pairwise non-isomorphic finite-dimensional indecomposable B-modules. Here, B is finite-dimensional and of finite representation type, thus every B-module is a direct sum of finite-dimensional indecomposables, see [2]. Then, we have:

- For any pregeneric $\mathcal{A}$-module G with $\mathrm{endol}(G) \leq d'$, the module $R(G)$ is isomorphic in B-Mod to a direct sum of modules in $\mathcal{I}(d')$.
- For any indecomposable $\mathcal{A}$-module M with $\dim_k M \leq d'$, the module $R(M)$ is isomorphic in B-Mod to a direct sum of modules in $\mathcal{I}(d')$.

In any case, B of finite representation type or not, let $X_1, \ldots, X_t$ be a complete set of pairwise non-isomorphic representatives of the B-modules in $\mathcal{I}(d')$ and make $X := X_1 \oplus \cdots \oplus X_t$. Consider the reduction $\mathcal{A} \mapsto \mathcal{A}^X$ described in [4], (7.3) and its associated functor $F_X : \mathcal{A}^X\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$. From [4], (2.7), the ditalgebra $\mathcal{A}^X$ is pregenerically tame and constructible. Then,

(a_1) The class of pregeneric $\mathcal{A}$-modules of endolength $\leq d'$ is contained in the union of the classes $\mathrm{Imgen}\, E$ and $\mathrm{Im}\, F_X$, where $\mathrm{Imgen}\, E$ denotes the class of $\mathcal{A}$-modules G of the form $G \cong E(N)$, for some pregeneric $N \in B$-Mod, and $\mathrm{Im}\, F_X$ denotes the indecomposable $\mathcal{A}$-modules G, which satisfy that $R(G) \cong \bigoplus_{i=1}^t \bigoplus_{I_i} X_i$ in B-Mod, for some index sets $I_1, \ldots, I_t$.

(a_2) The class of indecomposable $\mathcal{A}$-modules M with $\dim_k M \leq d'$ is contained in the union of the classes $\mathrm{Im}\, E$ and $\mathrm{Im}\, F_X$, where $\mathrm{Im}\, E$ denotes the class of $\mathcal{A}$-modules

M of the form $M \cong E(N)$, for some $N \in B$-mod, and $\operatorname{Im} F_X$ denotes the indecomposable $\mathcal{A}$-modules M, which satisfy that $R(M) \cong \bigoplus_{i=1}^{t} n_i X_i$ in B-Mod, for some $n_1, \dots, n_t \geq 0$.

From [4], (7.3)(2) we obtain:

(b_1) An $\mathcal{A}$-module G with the last property in (a_1) has the form $F_X(N) \cong G$, for some pregeneric $N \in \mathcal{A}^X$-Mod, see 4.4(1). Moreover, any sincere pregeneric $\mathcal{A}$-module G with $4\|G\|^e \leq d$ lies in $\operatorname{Imgen} E \cup \operatorname{Im} F_X$. This follows from the previous discussion and 8.5.
(b_2) An $\mathcal{A}$-module M with the last property in (a_2) has the form $F_X(N) \cong M$, for some $N \in \mathcal{A}^X$-mod. Moreover, any sincere indecomposable $\mathcal{A}$-module M with $\|M\| \leq d$ lies in $\operatorname{Im} E \cup \operatorname{Im} F_X$. This follows from the previous discussion using [4], (7.2) and the fact that $\dim_k M \leq 4\ell(M)$.

Assume that $\mathcal{A}^X$ is $(d-1)$-trivial. Then, it is sincerely $(d-1)$-trivial, and there are no sincere pregeneric $\mathcal{A}$-modules $G \in \operatorname{Im} F_X$ with $4\|G\|^e \leq d$, and there are only finitely many pairwise non-isomorphic sincere indecomposable $\mathcal{A}$-modules M with $\|M\| \leq d$. Then:

(c_1) Every sincere pregeneric $\mathcal{A}$-module G with $4\|G\|^e \leq d$ lies in $\operatorname{Imgen} E$. In particular, we have in this case that $\mathcal{A}$ admits exactly one pregeneric $\mathcal{A}$-module with $4\|G\|^e \leq d$.
(c_2) Almost every sincere indecomposable $\mathcal{A}$-module M with $\|M\| \leq d$ lies in $\operatorname{Im} E$.

Then, since $\mathcal{A}$ is not sincerely d-trivial, either B admits a pregeneric B-module or an infinite family of pairwise non-isomorphic finite-dimensional indecomposable B-modules. Thus, the algebra B has infinite representation type (if B is finite-dimensional, this follows for instance from [10], (7.3)).

Let us see that, in this case, the family $\mathcal{F} := \{F\}$ satisfies properties *1, 2', 3–5* for $\mathcal{A}$ and d.

We already know that F satisfies item *1*. Since there is only one sincere pregeneric $\mathcal{A}$-module G with $4\|G\|^e \leq d$ and it lies in $\operatorname{Imgen} E$, it has the form $G \cong E(N)$, for the pregeneric B-module N. From our Claim, we have $B = eBe$ and $F(N) \cong G$. Thus, item *2'* holds. Item *3* also holds clearly. From our Claim, the functor F maps indecomposable regular modules onto sincere $\mathcal{A}$-modules (item *4*). Finally, item *5* holds trivially because there is only one functor in $\mathcal{F}$.

From now on, we assume that $\mathcal{A}^X$ is not $(d-1)$-trivial.

Consider the constructible ditalgebras $\mathcal{A}^{Xd_1}, \dots, \mathcal{A}^{Xd_s}$ obtained from $\mathcal{A}^X$ by deletion of a finite number of idempotents of S, and the corresponding reduction functors F^{d_i} : $\mathcal{A}^{Xd_i}$-Mod $\longrightarrow \mathcal{A}^X$-Mod, for $i \in [1, s]$.

Since $\mathcal{A}^X$ is not $(d-1)$-trivial, we have the following two non-excluding possibilities:

(p_1) The ditalgebra $\mathcal{A}^X$ admits pregeneric modules N with $4\|N\|^e \leq d-1$. These pregeneric modules determine either sincere pregeneric $\mathcal{A}^X$-modules N with $4\|N\|^e \leq d-1$, thus $\mathcal{A}^X$ is not sincerely $(d-1)$-trivial, or there is a finite subset of idempotents of S such that the ditalgebra $\mathcal{A}^{Xd_i}$ obtained from $\mathcal{A}^X$ by eliminating these idempotents admits sincere pregeneric $\mathcal{A}^{Xd_i}$-modules N' with $F^{d_i}(N') \cong N$ and $4\|N'\|^e \leq d-1$.

(p_2) The ditalgebra $\mathcal{A}^X$ admits infinite families $\{N_t\}_t$ of pairwise non-isomorphic modules N_t with $\|N_t\| \leq d-1$. These modules determine either sincere indecomposable $\mathcal{A}^X$-modules N_t with $\|N_t\| \leq d-1$, thus $\mathcal{A}^X$ is not sincerely $(d-1)$-trivial, or there is a finite subset of idempotents of S such that the ditalgebra $\mathcal{A}^{Xd_i}$ obtained from $\mathcal{A}^X$ by eliminating these idempotents admits a family $\{N'_t\}_t$ of sincere indecomposable $\mathcal{A}^{Xd_i}$-modules N'_t with $F^{d_i}(N'_t) \cong N_t$ and $\|N'_t\| \leq d-1$.

Define $\mathcal{A}^{Xd_0} := \mathcal{A}^X$ and denote by $F^{d_0} : \mathcal{A}^{Xd_0}\text{-Mod} \longrightarrow \mathcal{A}^X\text{-Mod}$ the identity functor. Every sincere $\mathcal{A}^X$-module G lies in $\mathcal{A}^{Xd_0}$-Mod and $F^{d_0}(G) = G$.

Now, we consider the subset I of $[0,s]$ defined by $i \in I$ iff the ditalgebra $\mathcal{A}^{Xd_i}$ is not sincerely $(d-1)$-trivial. Thus $I \neq \emptyset$. Notice that, for $j \in [0,s] \setminus I$, the ditalgebra $\mathcal{A}^{Xd_j}$ admits only a finite number of isomorphism classes of sincere indecomposable modules N' with $\|N'\| \leq d-1$ and it admits no pregeneric module G' with $4\|G'\|^e \leq d-1$.

Then, apply the induction hypothesis to each $\mathcal{A}^{Xd_i}$ and $d-1$, for $i \in I$, to obtain minimal algebras $\{B_{ij}\}_{j=1}^{n_i}$ and functors $\{F_{ij} : B_{ij}\text{-Mod} \longrightarrow \mathcal{A}^{Xd_i}\text{-Mod}\}_{j=1}^{n_i}$ satisfying the corresponding requirements. Then, for any i and j, we can consider the compositions

$$B_{ij}\text{-Mod} \xrightarrow{F_{ij}} \mathcal{A}^{Xd_i}\text{-Mod} \xrightarrow{F^{d_i}} \mathcal{A}^X\text{-Mod} \xrightarrow{F_X} \mathcal{A}\text{-Mod}.$$

We will extract the family of functors we need for $\mathcal{A}$ and d from the family

$$\mathcal{F} := \{F\} \cup \{F^X F^{d_i} F_{ij} \mid i \in I \text{ and } j \in [1,n_i]\}.$$

First, we show that the functors in the family $\mathcal{F}$ cover every sincere pregeneric $\mathcal{A}$-module G with $4\|G\|^e \leq d$ and almost every sincere indecomposable $\mathcal{A}$-module M with $\|M\| \leq d$. That is items *2'* and *3* are satisfied by this family.

Indeed, given such a module G, by (b_1) in the above discussion, we have that $G \in \operatorname{Im} F_X \cup \operatorname{Imgen} E$. If G is a sincere pregeneric $\mathcal{A}$-module with $4\|G\|^e \leq d$, such that for some $N \in \mathcal{A}^X$-Mod, we have $F^X(N) \cong G$, then there is $i \in I$ and a sincere pregeneric $\mathcal{A}^{Xd_i}$-module L with $F^{d_i}(L) \cong N$, hence $4\|L\|^e = 4\|N\|^e < 4\|G\|^e \leq d$. Thus, $4\|L\|^e \leq d-1$. By the induction hypothesis, we get $L \cong F_{ij}(H)$, where H is the generic B_{ij}-module. Hence $F^X F^{d_i} F_{ij}(H) \cong G$, as claimed. If $G \in \operatorname{Imgen} E$, thus $G \cong E(N)$, for some generic $N \in B$-Mod. Since N is indecomposable and G is sincere, we have $B = eBe$, and $F(N) \cong G$.

Similarly, the functors in the family $\mathcal{F}$ cover almost every sincere indecomposable $\mathcal{A}$-module M with $\|M\| \leq d$. That is item *3* is satisfied by this family. Indeed, given such module M, by (b_2) in the above discussion, we have that $M \in \operatorname{Im} F_X \cup \operatorname{Im} E$. Moreover, for every such indecomposable M in $\operatorname{Im} F_X$, we have that $M \cong F_X(N)$ for some $N \in \mathcal{A}^X$-mod and $N \cong F^{d_i}(L)$, for some $i \in I$ and some sincere indecomposable $L \in \mathcal{A}^{Xd_i}$-Mod. We know that $\|L\| = \|N\| < \|M\| \leq d$, because M is sincere. By the induction hypothesis, for almost every such module L, we get $L \cong F_{ij}(H)$, where $H \in B_{ij}$-mod. Hence $F^X F^{d_i} F_{ij}(H) \cong M$, as claimed. If $M \in \operatorname{Im} E$, thus $M \cong E(N)$, for some $N \in B$-mod. Since N is indecomposable and M is sincere, we have $B = eBe$, and $F(N) \cong M$.

In the following discussion we will discard some functors of the family $\mathcal{F}$, without spoiling the covering conditions we have just proved for $\mathcal{F}$.

First, if B is of finite representation type then any pregeneric $G \in \mathcal{A}$-Mod has $R(G)$ isomorphic in B-Mod to a direct sum of modules in $\mathcal{I}(d')$, therefore $G \in \operatorname{Im} F^X$. Then, as we have just seen, every sincere pregeneric $\mathcal{A}$-module G in $\operatorname{Im} F_X$ with $4\|G\|^e \leq d$ has the form $F^X F^{d_i} F_{ij}(H) \cong G$, where H is the generic B_{ij}-module. Similarly, any indecomposable $M \in \mathcal{A}$-Mod has $R(M)$ isomorphic in B-Mod to a direct sum of modules in $\mathcal{I}(d')$, therefore $M \in \operatorname{Im} F^X$. Then, as before, almost every sincere indecomposable $\mathcal{A}$-module M in $\operatorname{Im} F_X$ with $\|M\| \leq d$ has the form $F^X F^{d_i} F_{ij}(H) \cong M$, for some $H \in B_{ij}$-mod. Thus we can discard the functor F from the family $\mathcal{F}$.

So, we will assume that the functor F is left in $\mathcal{F}$ only if B has infinite representation type. Then, the algebra eBe, as well as any of the minimal algebras B_{ij} are generically tame of infinite representation type.

If the functor F maps one indecomposable regular eBe-module H onto a non-sincere $\mathcal{A}$-module, from our previous Claim, we know that it maps each indecomposable eBe-module onto a non-sincere $\mathcal{A}$-module. In particular, the functor F maps the generic eBe-module onto a non-sincere pregeneric $\mathcal{A}$-module. In this case, again we can discard the functor F from the family $\mathcal{F}$.

So, we will assume that F is left in the family $\mathcal{F}$ only if eBe is such that F maps indecomposable regular eBe-modules onto sincere $\mathcal{A}$-modules.

Now, assume that the functor $F^X F^{d_i} F_{ij}$ is such that $F^X F^{d_i} F_{ij}(H)$ is not sincere, for some indecomposable regular B_{ij}-module H. Again, from [4], (6.6), we know the existence of a vector $\underline{\lambda}$ such that, for any indecomposable regular B_{ij}-module H', there is $c_{H'} \in \mathbb{N}$ with $\underline{\ell}(H') = c_{H'}\underline{\lambda}$. By 10.2, we get $\underline{\ell}^e(H_{ij}) = c\underline{\lambda}$, where H_{ij} is the generic B_{ij}-module. Having in mind 10.1, we have that $F^X F^{d_i} F_{ij}(H)$ is sincere if and only if $F^X F^{d_i} F_{ij}(H_{ij})$ is so. Hence, using [4], (7.4)(2), we can discard the functor $F^X F^{d_i} F_{ij}$ from our family $\mathcal{F}$.

So, we assume that the functor $F^X F^{d_i} F_{ij}$ appears in the family $\mathcal{F}$ only if it maps indecomposable regular modules onto sincere $\mathcal{A}$-modules.

Now, we have to show that the family $\mathcal{F}$, after discarding the functors pointed out above, satisfies items *1, 2', 3–5*. We already know that item *1* holds, because each functor in $\mathcal{F}$ is either a composition of reduction functors or it is F. Item *2'* and *3* hold, because we only discarded functors when we could cover every sincere pregeneric $\mathcal{A}$-module G such that $4\|G\|^e \leq d$ and almost every sincere indecomposable $\mathcal{A}$-module M such that

$\|M\| \leq d$ with the remaining functors in $\mathcal{F}$. Item *4* holds, because we discarded every functor in $\mathcal{F}$ without this property. In the following, we proceed to the proof of item *5*.

Notice first that, as a consequence of [4], (7.3)(2), if $L \in B$-Mod is an indecomposable such that $E(L) \cong F^X(L')$, for some $L' \in \mathcal{A}^X$-Mod, then $L \cong RE(L)$ has to be isomorphic to one of the indecomposable B-modules $X_1, \ldots, X_n$. Thus, for almost every indecomposable $N \in B$-Mod, there is no $L' \in \mathcal{A}^X$-Mod with $F^X(L') \cong E(N)$. This implies that there is no pair of infinite families of pairwise non-isomorphic indecomposables $\{N_u\}_{u\in U}$ in B-Mod and $\{M_u\}_{u\in U}$ in B_{ij}-Mod such that $F^X F^{d_i} F_{ij}(M_u) \cong F(N_u)$, for all $u \in U$.

Assume then that there is a pair of infinite families of pairwise non-isomorphic indecomposable regular modules $\{M_u\}_{u\in U}$ in B_{ij}-mod and $\{N_u\}_{u\in U}$ in $B_{i'j'}$-mod such that $F^X F^{d_{i'}} F_{i'j'}(N_u) \cong F^X F^{d_i} F_{ij}(M_u)$, for all $u \in U$. Then, since F^X reflects isomorphisms, we get $F^{d_{i'}} F_{i'j'}(N_u) \cong F^{d_i} F_{ij}(M_u)$, for all $u \in U$. In particular, they have the same support. But item *4* holds for the families $\{F_{ij}\}_j$ and $\{F_{i'j'}\}_{j'}$ and so $F_{i'j'}(N_u)$ and $F_{ij}(M_u)$ are sincere modules over $\mathcal{A}^{Xd_{i'}}$ and $\mathcal{A}^{Xd_i}$, respectively, hence $i = i'$. Then, $F_{ij'}(N_u) \cong F_{ij}(M_u)$, for all $u \in U$. From the induction hypothesis, $j = j'$.

Then, the family of functors $\mathcal{F}$ is what we wanted to construct, the last item follows from the given construction. □

Theorem 10.5. *Assume that an admissible ditalgebra $\mathcal{A}$ is constructible from a generically tame finite-dimensional basic algebra over a real closed field k. Then, for any integer $d \geq 0$, there are constructible ditalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_m$, generically tame minimal algebras of infinite representation type $B_1, \ldots, B_m$, where each B_i is an initial subalgebra of $\mathcal{A}_i$, and a family of functors $F_1, \ldots, F_m$ such that:*

1. *The functor $F_i : B_i$-Mod $\longrightarrow \mathcal{A}$-Mod preserves indecomposability and isomorphism classes, for any $i \in [1, m]$.*
2. *For each pregeneric $G \in \mathcal{A}$-Mod with $\mathrm{endol}(G) \leq d$ there is a unique $i \in [1, m]$ such that $F_i(H_i) \cong G$ in $\mathcal{A}$-Mod, where H_i denotes the unique generic B_i-module.*
3. *For almost every indecomposable $M \in \mathcal{A}$-Mod with $\dim_k M \leq d$ there are $i \in [1, m]$ and $N \in B_i$-mod such that $F_i(N) \cong M$ in $\mathcal{A}$-Mod.*
4. *If $\{N_u\}_{u\in U}$ and $\{M_u\}_{u\in U}$ are infinite families of pairwise non-isomorphic indecomposable regular modules in B_i-mod and B_j-mod, respectively, such that $F_i(N_u) \cong F_j(M_u)$ for all $u \in U$, then $i = j$.*
5. *Each functor F_i is the composition B_i-Mod $\xrightarrow{E_i} \mathcal{A}_i$-Mod $\xrightarrow{G_i} \mathcal{A}$-Mod, where E_i is the associated extension functor and G_i is the composition of the reduction functors associated to a finite sequence of reductions which transform $\mathcal{A}$ into $\mathcal{A}_i$.*

Proof. This proof is an adaptation of the poof of [4], (7.6). From [4], (4.6), we know that $\mathcal{A}$ is pregenerically tame. Notice that the statement of our Theorem 10.5 follows from the statement 10.5', obtained from 10.5 replacing items *2* and *3* by the following:

2'. *For each pregeneric $G \in \mathcal{A}$-Mod with $\|G\|^e \leq d$ there is a unique $i \in [1,m]$ such that $F_i(H_i) \cong G$ in $\mathcal{A}$-Mod, where H_i denotes the unique generic B_i-module.*

3'. *For almost every indecomposable $M \in \mathcal{A}$-Mod with $\|M\| \leq d$ there are $i \in [1,m]$ and $N \in B_i$-mod such that $F_i(N) \cong M$ in $\mathcal{A}$-Mod.*

Indeed, given $d \geq 0$, there are only finitely many endolength vectors $\underline{\ell}^e$ such that $\sum_{i=1}^n \ell_i^e \leq d$, consider their maximal endonorm $\hat{d} := \max_{\underline{\ell}^e}\{\|\underline{\ell}^e\|^e\}$. Then, $4\hat{d}$ is a non-negative integer and if *2'* holds for $4\hat{d}$, any pregeneric $\mathcal{A}$-module G with $\mathrm{endol}(G) \leq d$ has endolength vector $\underline{\ell}^e := \underline{\ell}^e(G)$ satisfying $\sum_{i=1}^n \ell_i^e \leq d$ and, therefore, $\|G\|^e \leq 4\hat{d}$. Similarly, given $d \geq 0$, there is a number $\tilde{d}$ such that for any indecomposable $\mathcal{A}$-module M with $\dim_k M \leq d$, we have $\|M\| \leq \tilde{d}$. Then, we can apply *2'* and *3'* to $d' := \max\{4\hat{d}, \tilde{d}\}$ to obtain *2* and *3*.

In this proof we shall say that an admissible ditalgebra $\mathcal{A}$ is *d-trivial* iff there is only a finite number of isoclasses of indecomposable $\mathcal{A}$-modules M with $\|M\| \leq d$, and there is no pregeneric $\mathcal{A}$-module G with $\|G\|^e \leq d$.

We assume that $\mathcal{A}$ is not d-trivial, otherwise, there is nothing to prove (the empty family of functors works).

Consider the admissible constructible ditalgebras $\mathcal{A}^{d_1}, \ldots, \mathcal{A}^{d_t}$ obtained from $\mathcal{A}$ by deletion of a finite number of idempotents of R. We consider also $\mathcal{A}^{d_0} := \mathcal{A}$ and the identity functor $F^{d_0} : \mathcal{A}^{d_0}\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$. Consider the subset I of $[0,t]$ defined by $i \in I$ iff $\mathcal{A}^{d_i}$ is not d-trivial. Discard all the ditalgebras $\mathcal{A}^{d_i}$ with $i \notin I$.

Then, apply 10.4 to each $\mathcal{A}^{d_i}$ and d, for $i \in I$, to obtain minimal algebras $\{B_{ij}\}_{j=1}^{n_i}$ and functors $F_{ij} : B_{ij}\text{-Mod} \longrightarrow \mathcal{A}^{d_i}\text{-Mod}$ satisfying the corresponding statements *1–6* of 10.4 for each $\mathcal{A}^{d_i}$ and d. Then, we can consider the family of compositions

$$\mathcal{F} := \left\{B_{ij}\text{-Mod} \xrightarrow{F_{ij}} \mathcal{A}^{d_i}\text{-Mod} \xrightarrow{F^{d_i}} \mathcal{A}\text{-Mod} \;\middle|\; i \in I \text{ and } j \in [1,n_i]\right\}.$$

It is clear that the family $\mathcal{F}$ satisfies item *1*, because the families $\{F_{ij}\}_j$ do so and F^{d_i} preserves indecomposables and isomorphism classes. The family $\mathcal{F}$ also satisfies *2'* because, given any pregeneric $G \in \mathcal{A}$-Mod with $\|G\|^e \leq d$, we have $G \cong F^{d_i}(N)$, for some sincere pregeneric $N \in \mathcal{A}^{d_i}$-Mod with $\|N\|^e = \|G\|^e \leq d$. For each one of these pregeneric modules N, we have $F_{ij}(H_{ij}) \cong N$, where H_{ij} denotes the generic B_{ij}-module. In a similar way, one shows that $\mathcal{F}$ satisfies item *3'*.

Assume then that there is a pair of infinite families of pairwise non-isomorphic indecomposable regular modules $\{N_u\}_{u\in U}$ in B_{ij}-mod and $\{M_u\}_{u\in U}$ in $B_{i'j'}$-mod such that $F^{d_{i'}}F_{i'j'}(M_u) \cong F^{d_i}F_{ij}(N_u)$, for all $u \in U$. Then, proceeding as in the proof of the last theorem, we obtain $i = i'$ and $j = j'$. □

Theorem 10.6. *Assume that an admissible ditalgebra $\mathcal{A}$ is constructible from a generically tame finite-dimensional basic algebra over a real closed field k. Fix $d \in \mathbb{N}$. Then, there are constructible ditalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_m$, generically tame minimal algebras of infinite* *representation type $B_1, \dots, B_m$, where each B_i is an initial subalgebra of $\mathcal{A}_i$, principal ideal domains $\Gamma_1, \dots, \Gamma_m \in \mathcal{R}$, and a family of functors $\hat{F}_1, \dots, \hat{F}_m$ satisfying that:*

1. *The functor $\hat{F}_i : \Gamma_i\text{-Mod} \longrightarrow \mathcal{A}\text{-Mod}$ preserves indecomposability and isomorphism classes, for any $i \in [1, m]$.*
2. *For each pregeneric module $G \in \mathcal{A}\text{-Mod}$ with $\operatorname{endol}(G) \leq d$ there is a unique $i \in [1, m]$ such that $\hat{F}_i(Q_i) \cong G$ in $\mathcal{A}\text{-Mod}$, where Q_i denotes the unique generic Γ_i-module.*
3. *For almost every indecomposable $M \in \mathcal{A}\text{-Mod}$ with $\dim_k M \leq d$ there are $i \in [1, m]$ and $N \in \Gamma_i\text{-mod}$ such that $\hat{F}_i(N) \cong M$ in $\mathcal{A}\text{-Mod}$.*
4. *If $\{N_u\}_{u \in U}$ and $\{M_u\}_{u \in U}$ are infinite families of pairwise non-isomorphic indecomposable modules in $\Gamma_i\text{-mod}$ and $\Gamma_j\text{-mod}$, respectively, such that $\hat{F}_i(N_u) \cong \hat{F}_j(M_u)$, for all $u \in U$, then $i = j$.*
5. *Each functor $\hat{F}_i$ is given by the composition*

$$\Gamma_i\text{-Mod} \xrightarrow{H_i} B_i\text{-Mod} \xrightarrow{E_i} \mathcal{A}_i\text{-Mod} \xrightarrow{F_i} \mathcal{A}\text{-}\operatorname{Mod},$$

where E_i is the associated extension functor, F_i is the composition of reduction functors associated to a finite sequence of reductions which transform $\mathcal{A}$ into $\mathcal{A}_i$, the algebra Γ_i associated to B_i and the functor $H_i = Y_i \otimes_{\Gamma_i} - : \Gamma_i\text{-Mod} \longrightarrow B_i\text{-Mod}$ are those described in 9.7.

Proof. It follows immediately from 10.5 and 9.7. For item *4*, one has to remember that, if B_i is finite-dimensional, the indecomposable regular B_i-modules are characterized by the fact that their length vector is a multiple of the generator of the radical of the quadratic form of the algebra. Thus, the image of H_i consists of regular B_i-modules, see 9.6. □

Corollary 10.7. *Assume that an admissible ditalgebra $\mathcal{A}$ is constructible from a generically tame finite-dimensional basic algebra over a real closed field k. Then, for any pregeneric $\mathcal{A}$-module G, there is a realization Z of G, over some algebra $\Gamma \in \mathcal{R}$, which is free as a right Γ-module and such that the composition*

$$\Gamma\text{-Mod} \xrightarrow{Z \otimes_\Gamma -} A\text{-Mod} \xrightarrow{L_{\mathcal{A}}} \mathcal{A}\text{-Mod}$$

preserves indecomposables and isomorphism classes. Moreover, if Q denotes the skew field of fractions of Γ, then $\operatorname{End}_{\mathcal{A}}(G)/\operatorname{rad}\operatorname{End}_{\mathcal{A}}(G) \cong Q$.

Proof. If G is a pregeneric $\mathcal{A}$-module with endolength d, then applying 10.6, we obtain a constructible ditalgebra $\mathcal{A}_i$, a generically tame minimal algebra of infinite representation type B_i, a principal ideal domain $\Gamma_i \in \mathcal{R}$, and a functor $\hat{F}_i$ defined as the composition:

$$\Gamma_i\text{-Mod} \xrightarrow{H_i} B_i\text{-Mod} \xrightarrow{E_i} \mathcal{A}_i\text{-Mod} \xrightarrow{F_i} \mathcal{A}\text{-}\operatorname{Mod},$$

as described in 10.6, such that $G \cong \hat{F}_i(Q_i)$, where Q_i is the generic Γ_i-module.

Then, the same argument given in the proof of 9.10 shows that $Z = F_i E_i(Y_i)$, where Y_i is the B_i-Γ_i-bimodule defined in 9.7, is the wanted realization for G over the algebra $\Gamma = \Gamma_i$. □

11. Transition to finite-dimensional algebras

In this section we transfer our results from modules over constructible ditalgebras to modules over finite-dimensional algebras. We provide detailed proofs which follow the usual strategy.

Remark 11.1. Given finite-dimensional algebras Λ and Γ, and an equivalence functor $F : \Lambda\text{-Mod} \longrightarrow \Gamma\text{-Mod}$, it is well known that F preserves the property of having finite endolength and that there is a positive integer b_F such that

$$\operatorname{endol}\big(F(M)\big) \leq b_F \times \operatorname{endol}(M), \quad \text{for any } M \in \Lambda\text{-Mod}.$$

This was used by Crawley-Boevey in [9]. We include here a simple proof of this fact (the statement [6], (29.8)(1) is incorrect).

By Morita's Theorem, we may assume that $F = P \otimes_\Lambda -$, for some Γ-Λ-bimodule P which is finitely generated projective as a Λ-module. Then, there is an epimorphism $\Lambda^b \longrightarrow P$ of right Λ-modules. Assume that $M \in \Lambda$-Mod has finite endolength and make $E := \operatorname{End}_\Lambda(M)^{op}$. Then, we have an epimorphism $\Lambda^b \otimes_\Lambda M \longrightarrow P \otimes_\Lambda M$ of right E-modules. Moreover, since F is full and faithful, we get the first equality in the following: $\operatorname{endol}(P \otimes_\Lambda M) = \ell_E(P \otimes_\Lambda M) \leq b \times \ell_E(M) = b \times \operatorname{endol}(M)$.

Similarly, one shows that $\dim_k F(N) \leq b_F \times \dim_k N$, for any $N \in \Lambda$-Mod.

Theorem 11.2. *Let Λ be a generically tame finite-dimensional algebra over a real closed field k and let d be a non-negative integer. Then, there is a finite sequence of principal ideal domains $\Gamma_1, \ldots, \Gamma_m \in \mathcal{R}$, and Λ-Γ_i-bimodules $Z_1, \ldots, Z_m$, which are finitely generated as right Γ_i-modules, satisfying the following:*

1. *The functor $U_i := Z_i \otimes_{\Gamma_i} - : \Gamma_i\text{-Mod} \longrightarrow \Lambda\text{-Mod}$ preserves indecomposables, generic modules, and isomorphism classes. Moreover, if Q_i is the skew field of fractions of Γ_i, then $\operatorname{End}_\Lambda(U_i(Q_i))/\operatorname{rad}\operatorname{End}_\Lambda(U_i(Q_i)) \cong Q_i$.*
2. *For each generic Λ-module G with $\operatorname{endol}(G) \leq d$ there is a unique $i \in [1, m]$ such that $G \cong Z_i \otimes_{\Gamma_i} Q_i$, where Q_i is the generic Γ_i-module. Moreover, Z_i is a realization of G over Γ_i.*
3. *Almost every indecomposable Λ-module M with $\dim_k M \leq d$ is isomorphic to $Z_i \otimes_{\Gamma_i} N$, for some $i \in [1, m]$ and some $N \in \Gamma_i$-mod.*
4. *If $\{N_u\}_{u \in U}$ and $\{M_u\}_{u \in U}$ are infinite families of pairwise non-isomorphic indecomposable modules in Γ_i-mod and Γ_j-mod, respectively, such that $Z_i \otimes_{\Gamma_i} N_u \cong Z_j \otimes_{\Gamma_j} M_u$ for all $u \in U$, then $i = j$.*

Proof. We first show that we can assume that Λ is a basic algebra. Indeed, assume the theorem holds for basic algebras, assume Λ is not basic and take $d \geq 0$. Consider a basic finite-dimensional k-algebra Λ' which is Morita equivalent to Λ. Then, we know that Λ' is generically tame. Consider a Λ-Λ'-bimodule P, finitely generated projective by the right which realizes the equivalence $P \otimes_{\Lambda'} - : \Lambda'\text{-Mod} \longrightarrow \Lambda\text{-Mod}$. Call $\Theta : \Lambda\text{-Mod} \longrightarrow \Lambda'\text{-Mod}$ its quasi inverse. From 11.1, this equivalence preserves generic modules and there is a constant b_Θ such that for any generic Λ-module G we have $\text{endol}(\Theta(G)) \leq b_\Theta \times \text{endol}(G)$, and $\dim_k \Theta(M) \leq b_\Theta \times \dim_k M$, for any $M \in \Lambda\text{-mod}$.

Make $d' := b_\Theta \times d$. Then, by assumption, there are algebras $\Gamma_1, \ldots, \Gamma_m \in \mathcal{R}$ and Λ'-Γ_i-bimodules $Z'_1, \ldots, Z'_m$, which are finitely generated as right Γ_i-modules, satisfying the following:

1'. *The functor* $U'_i := Z'_i \otimes_{\Gamma_i} - : \Gamma_i\text{-Mod} \longrightarrow \Lambda'\text{-Mod}$ *preserves indecomposables, generic modules, and isomorphism classes. Moreover, if* Q_i *is the skew field of fractions of* Γ_i*, then* $\text{End}_{\Lambda'}(U'_i(Q_i))/\operatorname{rad}\text{End}_{\Lambda'}(U'_i(Q_i)) \cong Q_i$.
2'. *For each generic* Λ'*-module* G' *with* $\text{endol}(G') \leq d'$*, there is a unique* $i \in [1, m]$ *such that* $G' \cong Z'_i \otimes_{\Gamma_i} Q_i$*, where* Q_i *is the generic* Γ_i*-module. Moreover,* Z'_i *is a realization of* G' *over* Γ_i.
3'. *Almost every indecomposable* Λ'*-module* M *with* $\dim_k M \leq d'$ *is isomorphic to* $Z'_i \otimes_{\Gamma_i} N$*, for some* $i \in [1, m]$ *and some* $N \in \Gamma_i\text{-mod}$.
4'. *If* $\{N_u\}_{u \in U}$ *and* $\{M_u\}_{u \in U}$ *are infinite families of pairwise non-isomorphic indecomposable modules in* $\Gamma_i\text{-mod}$ *and* $\Gamma_j\text{-mod}$*, respectively, such that* $Z'_i \otimes_{\Gamma_i} N_u \cong Z'_j \otimes_{\Gamma_j} M_u$ *for all* $u \in U$*, then* $i = j$.

Consider, for each i, the Λ-Γ_i-bimodule $Z_i := P \otimes_{\Lambda'} Z'_i$. Since P is a finitely generated projective right Λ'-module, each bimodule Z_i is finitely generated as a right Γ_i-module. It follows that items *1* to *4* hold for Λ with the given bimodules $Z_1, \ldots, Z_m$. Moreover, from *2'* and 4.19, we know that Z_i is a realization of G over Γ_i in item *2*.

From now on, we assume that our given finite-dimensional algebra Λ is basic. Since k is a perfect field, the algebra Λ splits over its radical J and, therefore, its Drozd's ditalgebra $\mathcal{D} := \mathcal{D}^\Lambda$ is admissible. By definition, $\mathcal{D}$ is constructible from the generically tame algebra Λ. Consider the splitting $\Lambda = S \oplus J$ over the radical J of Λ. Apply 10.6 to the ditalgebra $\mathcal{D}$ and the integer $d' := (1 + \dim_k \Lambda) \dim_k \Lambda \times d$ to obtain the corresponding constructible ditalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_m$ and generically tame minimal algebras of infinite representation type $B_1, \ldots, B_m$, where each B_i is an initial subalgebra of $\mathcal{A}_i$, and the family of principal ideal domains $\Gamma_1, \ldots, \Gamma_m \in \mathcal{R}$, and the corresponding family of functors $\hat{F}_i : \Gamma_i\text{-Mod} \longrightarrow \mathcal{D}\text{-Mod}$ such that 10.6 (1)–(5) hold for $\mathcal{D}$ and d'.

For a fixed $i \in [1, m]$, adopt the notation $\mathcal{A} = \mathcal{A}_i = \mathcal{D}^{z_1 \cdots z_n}$. Then, we have that the functor $\hat{F}_i$ is isomorphic to the composition:

$$\Gamma_i\text{-Mod} \xrightarrow{H_i} B_i\text{-Mod} \xrightarrow{E_i} \mathcal{D}^{z_1 \cdots z_n}\text{-Mod} \xrightarrow{F_i = F^{z_1} \cdots F^{z_n}} \mathcal{D}\text{-Mod}.$$

Here $H_i = Y_i \otimes_{\Gamma_i} -$, where Y_i is a B_i-Γ_i-bimodule, free of finite rank as a right Γ_i-module, as in 9.7. Hence the A-Γ_i-bimodule $E_i(Y_i)$ is free finitely generated by the right. We have the equality of functors $L_{\mathcal{A}}(E_i(Y_i) \otimes_{\Gamma_i} -) = E_i(Y_i \otimes_{\Gamma_i} -)$ and we can apply [6], (22.7), to obtain that $F_i E_i(Y_i)$ is a D-Γ_i-bimodule projective by the right and the composition of the functor $E_i(Y_i) \otimes_{\Gamma_i} -$ with the restriction $A\text{-Mod} \longrightarrow D\text{-Mod}$ of F_i is given by the tensor $F_i E_i(Y_i) \otimes_{\Gamma_i} -$. Notice that $\hat{F}_i \cong L_{\mathcal{D}}(F_i E_i(Y_i) \otimes_{\Gamma_i} -)$ and recall that it preserves isomorphism classes, indecomposables, and pregeneric modules.

Consider the usual equivalence functor $\Xi_\Lambda : \mathcal{D}\text{-Mod} \longrightarrow \mathcal{P}^1(\Lambda)$ and, for $i \in [1, m]$, set $Z_i := Z \otimes_D F_i E_i(Y_i)$, where Z is the transition bimodule associated to Λ, as in [6], (22.18). Then, each Z_i is finitely generated over Γ_i by construction.

For each i, denote by U_i the composition

$$\Gamma_i\text{-Mod} \xrightarrow{\hat{F}_i} \mathcal{D}\text{-Mod} \xrightarrow{\Xi_\Lambda} \mathcal{P}^1(\Lambda) \xrightarrow{\text{Cok}} \Lambda\text{-Mod},$$

which is, by [6], (22.18), naturally isomorphic to

$$\operatorname{Cok} \Xi_\Lambda L_{\mathcal{D}}\big(F_i E_i(Y_i) \otimes_{\Gamma_i} -\big) \cong Z \otimes_D F_i E_i(Y_i) \otimes_{\Gamma_i} - = Z_i \otimes_{\Gamma_i} -.$$

The proof of the first part of (1), which uses [6], (22.20), is the same as the proof of the same item of [4], (10.2), so we skip the argument here.

We have $U_i(Q_i) \cong Z_i \otimes_{\Gamma_i} Q_i \cong \operatorname{Cok} \Xi_\Lambda \hat{F}_i(Q_i)$. As in the proof of 9.10, using 9.9 and [6], (31.6), we obtain

$$\begin{aligned} \operatorname{End}_\Lambda(U_i(Q_i))/\operatorname{rad}\operatorname{End}_\Lambda(U_i(Q_i)) &\cong \operatorname{End}_{\mathcal{D}}(\hat{F}_i(Q_i))/\operatorname{rad}\operatorname{End}_{\mathcal{D}}(\hat{F}_i(Q_i)) \\ &\cong \operatorname{End}_{\Gamma_i}(Q_i)/\operatorname{rad}\operatorname{End}_{\Gamma_i}(Q_i) \cong Q_i^{op} \cong Q_i. \end{aligned}$$

(2): Let G be a generic Λ-module with $\operatorname{endol}(G) \leq d$ and take $H \in \mathcal{D}\text{-Mod}$ with $\operatorname{Cok} \Xi_\Lambda(H) \cong G$. From [4], (4.4)(3), we obtain

$$\operatorname{endol}(H) \leq (1 + \dim_k \Lambda) \times \operatorname{endol}(G) \leq d'.$$

From 10.6, we know that for each such pregeneric $\mathcal{D}$-module H, we have that $H \cong \hat{F}_i(Q_i)$ in $\mathcal{A}$-Mod, for a unique $i \in [1, m]$, where Q_i is the generic Γ_i-module. Hence, $G \cong \operatorname{Cok} \Xi_\Lambda(H) \cong \operatorname{Cok} \Xi_\Lambda \hat{F}_i(Q_i) \cong Z_i \otimes_{\Gamma_i} Q_i$. As in the proof of 10.7, we know that $\hat{F}_i(\Gamma_i)$ is a realization of H over Γ_i. From 4.14, we get that Z_i is a realization of G over Γ_i.

(3): Similarly, from [4], (4.4)(2), if M is an indecomposable Λ-module with $\dim_k M \leq d$ and $L \in \mathcal{D}\text{-Mod}$ satisfies $\operatorname{Cok} \Xi_\Lambda(L) \cong M$, then $\dim_k L \leq d'$. From 10.6, we know that for almost every such modules L, we have that $L \cong \hat{F}_i(N)$, for some $i \in [1, m]$ and $N \in \Gamma_i\text{-mod}$. Hence, $M \cong \operatorname{Cok} \Xi_\Lambda(L) \cong \operatorname{Cok} \Xi_\Lambda \hat{F}_i(N) \cong Z_i \otimes_{\Gamma_i} N$.

(4): Assume that $\{N_u\}_{u \in U}$ and $\{M_u\}_{u \in U}$ are infinite families of pairwise non-isomorphic indecomposable modules in Γ_i-mod and Γ_j-mod, respectively, such that $\operatorname{Cok} \Xi_\Lambda \hat{F}_i(N_u) \cong \operatorname{Cok} \Xi_\Lambda \hat{F}_j(M_u)$, for all $u \in U$. We already know that $\Xi_\Lambda \hat{F}_i(N_u)$,

$\Xi_\Lambda \hat{F}_j(M_u) \in \mathcal{P}^2(\Lambda)$. Therefore, the existence of an isomorphism $\mathrm{Cok}\, \Xi_\Lambda \hat{F}_i(N_u) \cong \mathrm{Cok}\, \Xi_\Lambda \hat{F}_j(M_u)$ in Λ-Mod, together with [6], (18.10)(3), imply that $\hat{F}_i(N_u) \cong \hat{F}_j(M_u)$, for $u \in U$. From 10.6(4), we get $i = j$. □

Remark 11.3. Whenever Γ is a centrally bounded principal ideal domain, as in [4], (6.3), and f is a central non-zero element of Γ, the central localization Γ_f, on the multiplicative subset $\{f^i \mid i \geq 0\}$, is again a centrally bounded principal ideal domain, see [18], §1.10. Moreover, if $0 \neq f \in \Gamma$ is central, the morphism of Γ-modules $\mu_f : \Gamma/\Gamma p \longrightarrow \Gamma/\Gamma p$ induced by multiplication by f is invertible (that is non-zero, since $\Gamma/\Gamma p$ is simple) for any atom p not similar to any atom in the atomic decomposition of f in Γ, see [15], (1.2.9). Thus, almost every simple Γ-module is a simple Γ_f-module too. This implies that the finite-length indecomposable Γ_f-modules are obtained from those of Γ by elimination of a finite number of tubes, see [4], (6.5).

Theorem 11.4. *Let Λ be a generically tame finite-dimensional algebra over a real closed field k. Then,*

1. *For any generic Λ-module G, there is a realization Z of G over some algebra Γ_f, where $\Gamma \in \mathcal{R}$ and f is a non-zero central element of Γ, such that Z is a right Γ_f-module free of finite rank, and the functor $Z \otimes_{\Gamma_f} - : \Gamma_f\text{-Mod} \longrightarrow \Lambda\text{-Mod}$ preserves indecomposables and isomorphism classes.*
2. *For each $d \in \mathbb{N}$, there are generic Λ-modules $G_1, \ldots, G_m$ and, for each $i \in [1, m]$, a realization Z_i of G_i over some $\Gamma_i \in \mathcal{R}$, such that: for almost all indecomposable Λ-module M with $\dim_k M \leq d$, we have an isomorphism $M \cong Z_i \otimes_{\Gamma_i} N$, for some $i \in [1, m]$ and some indecomposable $N \in \Gamma_i\text{-mod}$.*

Proof. (1): Given a generic Λ-module G, say of endolength d, we can apply 11.2 to obtain a principal ideal domain $\Gamma \in \mathcal{R}$ and a Λ-Γ-bimodule Z' which is finitely generated as a right Γ-module such that $Z' \otimes_\Gamma - : \Gamma\text{-Mod} \longrightarrow \Lambda\text{-Mod}$ preserves indecomposability and isomorphism classes and $G \cong Z' \otimes_\Gamma Q$, where Q is the skew field of fractions of Γ. Moreover, Z' is a realization of G over Γ. Then, we have the exact sequence of Λ-Γ-bimodules $0 \to tZ' \to Z' \to Z'/tZ' \to 0$, where tZ' denotes the torsion submodule of the right Γ-module Z' and Z'/tZ' is a free right Γ-module of finite rank (say r). By [8], §8.2, 2.4, since tZ' is finitely generated, it is a finite direct sum of torsion cyclic modules. But Γ is a right Ore domain, therefore tZ' has a non-zero annihilator. Since Γ is a centrally bounded principal ideal domain, as in [4], (6.3), we know there is a non-zero central element $f \in \Gamma$ which annihilates tZ'. The right Γ_f-module $Z := Z' \otimes_\Gamma \Gamma_f \cong (Z'/tZ') \otimes_\Gamma \Gamma_f$ is free of rank r. Moreover, $G \cong Z' \otimes_\Gamma Q \cong Z' \otimes_\Gamma \Gamma_f \otimes_{\Gamma_f} Q = Z \otimes_{\Gamma_f} Q$ and, therefore, $\mathrm{endol}(G) = r = \dim_Q G$. Thus, Z is a realization of G over Γ_f. Finally, note that $Z \otimes_{\Gamma_f} -$ is the composition of $Z' \otimes_\Gamma -$ with $\Gamma_f \otimes_{\Gamma_f} -$, and both of them preserve indecomposables and isomorphism classes.

(2): Given $d \in \mathbb{N}$, apply 11.2 to obtain algebras $\Gamma_1, \ldots, \Gamma_m \in \mathcal{R}$ and, for each $i \in [1, m]$, a Λ-Γ_i-bimodule Z_i which is finitely generated as a right Γ_i-module such that 11.2(3) holds. For $i \in [1, m]$, make $G_i := Z_i \otimes_{\Gamma_i} Q_i$, where Q_i is the skew field of fractions of Γ_i. Then, from 11.2(1) and the proof of 11.2(2), the module G_i is generic and the bimodule Z_i is a realization for G_i over Γ_i, for each $i \in [1, m]$. Finally, we can apply 11.2(3), to finish our proof. □

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