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On restrictions of generic modules of tame algebras

Research Article

Raymundo Bautista[1*], Efrén Pérez[2†], Leonardo Salmerón[1‡]

1 *Centro de Ciencias Matemáticas, Universidad Nacional Autónoma de México, Morelia, 58089, México*

2 *Facultad de Matemáticas, Universidad Autónoma de Yucatán, Mérida, 97119, México*

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Abstract: Given a convex algebra Λ_0 in the tame finite-dimensional basic algebra Λ, over an algebraically closed field, we describe a special type of restriction of the generic Λ-modules.

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1. Introduction

In this article, we assume that the ground field k is algebraically closed. All our algebras Λ are associative k-algebras with unit element, and Λ-Mod denotes the category of (left) Λ-modules.

The following situation arises frequently in the representation theory of algebras. Let Λ be a finite-dimensional algebra and take any idempotent e_0 of Λ. If we make $\Lambda_0 = e_0\Lambda e_0$, we have the standard restriction functor $\rho\colon \Lambda\text{-Mod} \to \Lambda_0\text{-Mod}$, where $\rho(M) = e_0M$, for any $M \in \Lambda\text{-Mod}$. This functor admits as a left adjoint the functor $\mathrm{tens} = \Lambda e_0 \otimes_{\Lambda_0} -\colon \Lambda_0\text{-Mod} \to \Lambda\text{-Mod}$, which is full and faithful, see for instance [1, § I.6].

Let us recall some terminology from [2]. Given a finite-dimensional basic algebra Λ, over our algebraically closed field k, there is a semisimple subalgebra S of Λ such that Λ admits a decomposition $\Lambda = S \bigoplus \operatorname{rad}\Lambda$ of S-S-bimodules. Consider a decomposition $1 = \sum_{e\in E} e$ of the unit element of S as a sum of central primitive orthogonal idempotents in S, and let E_0 be a subset of E. Then, E_0 is called *convex* if and only if, whenever $e''\Lambda e'\Lambda e \neq 0$ with $e'', e \in E_0$ and $e' \in E$, we have that $e' \in E_0$.

* *E-mail: raymundo@matmor.unam.mx*
† *E-mail: jperezt@uady.mx*
‡ *E-mail: salmeron@matmor.unam.mx*

Given a convex subset E_0 of E, we are interested in the algebra $\Lambda_0 = e_0\Lambda e_0$, where $e_0 = \sum_{e\in E_0} e$, and we want to establish some relations between the categories Λ-mod and Λ_0-mod. Notice that Λ_0 is also a basic finite-dimensional algebra which splits over its radical: $\Lambda_0 = S_0 \bigoplus \operatorname{rad}\Lambda_0$, where $S_0 = e_0 S e_0$ and $\operatorname{rad}\Lambda_0 = e_0(\operatorname{rad}\Lambda)e_0$. The algebra Λ_0 is called *convex in* Λ if E_0 is a convex subset of E. Notice that our definition of convexity differs from the one commonly used in the theory of locally bounded categories.

Given a convex algebra Λ_0 in Λ, the morphism $\psi\colon \Lambda \to \Lambda_0$ given by $\psi(\lambda) = e_0\lambda e_0$, $\lambda \in \Lambda$, is a morphism of algebras. Therefore, we can consider the Λ_0-Λ-bimodule Λ_0 and a new type of natural restriction functor

$$\operatorname{res} = \Lambda_0 \otimes_\Lambda - \colon\ \Lambda\text{-Mod} \to \Lambda_0\text{-Mod}.$$

We denote by $\mathcal{P}(\Lambda)$ and $\mathcal{P}(\Lambda_0)$ the categories of morphisms between projective Λ-modules and projective Λ_0-modules, respectively. Then, the functors tens and res induce functors $\operatorname{Tens}\colon \mathcal{P}(\Lambda_0) \to \mathcal{P}(\Lambda)$ and $\operatorname{Res}\colon \mathcal{P}(\Lambda) \to \mathcal{P}(\Lambda_0)$ such that $\operatorname{res}\operatorname{Cok} \cong \operatorname{Cok}_0 \operatorname{Res}$ and $\operatorname{Cok}\operatorname{Tens} \cong \operatorname{tens}\operatorname{Cok}_0$, where $\operatorname{Cok}\colon \mathcal{P}(\Lambda) \to \Lambda\text{-Mod}$ and $\operatorname{Cok}_0\colon \mathcal{P}(\Lambda_0) \to \Lambda_0\text{-Mod}$ are the cokernel functors. Moreover,

$$\operatorname{res}\operatorname{tens} \cong 1_{\Lambda_0\text{-Mod}}$$

and, then, given $M \in \Lambda$-Mod, we have that $M \cong \operatorname{tens}\operatorname{res} M$ if and only if $M \cong \operatorname{tens} M'$, for some $M' \in \Lambda_0$-Mod, see [2]. We keep the notation introduced before for the rest of this paper.

Recall also that, given a Λ-module G, by definition, the *endolength* of G is its length as a right $\operatorname{End}_\Lambda(G)^{\mathrm{op}}$-module. The module G is called *generic* if it is indecomposable, of infinite length as a Λ-module, but with finite endolength. The algebra Λ is called *generically tame* if, for each $d \in \mathbb{N}$, there is only a finite number of isomorphism classes of generic Λ-modules of endolength d. This notion was introduced by Crawley-Boevey in [5], providing a new definition of tameness, which coincides with the usual notion of tameness for finite-dimensional algebras over algebraically closed fields, but which makes sense for arbitrary algebras.

We will show that given a convex algebra Λ_0 in the tame algebra Λ and any generic Λ-module G, the Λ_0-module $\operatorname{res} G$ has finite endolength and either it is generic, or it is a direct sum of some finite-dimensional Λ_0-modules. This is our Corollary 4.3; see also the more complete Theorem 4.2.

This theorem is proved using matrix problem methods and we resort to the ditalgebra language of [4]. It is obtained as a consequence of the discussion of parametric families of modules through realizations proposed by Crawley-Boevey in [5] and the study of extension/restriction interactions between module categories over a ditalgebra and a proper subditalgebra presented in [2].

Our Theorem 3.4 for ditalgebras has its own importance because it can be used to study relations of the generic modules with bounded endolength over a finite-dimensional algebra Λ, with the generic modules over hereditary algebras. This can be done in the same way that [2] was used in [3] to relate the corresponding finite-dimensional indecomposables with bounded dimension.

2. Families of modules

As usual, given any k-ditalgebra $\mathcal{A}$, we denote by $\mathcal{A}$-Mod the category of $\mathcal{A}$-modules, see [4, 2.2]. Recall from [3] the following definitions.

Definition 2.1.
Let $\mathcal{A}$ be a layered ditalgebra, with layer (R, W), see [4, § 4]. Given $M \in \mathcal{A}$-Mod, denote by $E_M = \operatorname{End}_{\mathcal{A}}(M)^{\mathrm{op}}$ its endomorphism algebra. Then, M admits a structure of R-E_M-bimodule, where $m \cdot (f^0, f^1) = f^0(m)$, for $m \in M$ and $(f^0, f^1) \in E_M$. By definition, the *endolength* of M, denoted by $\operatorname{endol} M$, is the length of M as a right E_M-module. A module $M \in \mathcal{A}$-Mod is called *pregeneric* if M is indecomposable, with finite endolength, but with infinite dimension over the ground field k.

If B is any k-algebra, we have the corresponding regular ditalgebra $\mathcal{B}$ with layer $(B, 0)$. Then, the categories $\mathcal{B}$-Mod and B-Mod can be identified canonically. If the algebra B is finite-dimensional, the notion of pregeneric $\mathcal{B}$-module coincides with the notion of generic B-module. For infinite-dimensional algebras, this is not always the case. A recurrent argument in the reduction techniques used to study modules over finite-dimensional algebras Λ, passes from the module category of a ditalgebra $\mathcal{A}$, after some reduction process, to the category of projective presentations, and then to Λ-Mod. This process maps pregeneric $\mathcal{A}$-modules onto generic Λ-modules, see [3].

Notation 2.2.

Throughout this work, given a ditalgebra $\mathcal{A} = (T, \delta)$, we denote with a roman A the subalgebra $[T]_0$ of degree zero elements of the underlying graded algebra T of $\mathcal{A}$, see [4, § 1]. Then, the categories A-Mod and $\mathcal{A}$-Mod share the same class of objects, but there are more morphisms in $\mathcal{A}$-Mod. There is a canonical embedding functor $L_{\mathcal{A}} \colon A\text{-Mod} \to \mathcal{A}\text{-Mod}$, which is the identity on objects and maps each $f^0 \in \mathrm{Hom}_A(M, N)$ onto $L_{\mathcal{A}}(f^0) = (f^0, 0)$.

It may be the case that for a layered ditalgebra $\mathcal{A}$, an $\mathcal{A}$-module M admits a non-trivial decomposition in $\mathcal{A}$-Mod but is indecomposable in A-Mod. Thus, it is not always true that the functor $L_{\mathcal{A}}$ preserves indecomposability.

Reminder 2.3.

In this work, we have to deal mainly with *seminested ditalgebras* $\mathcal{A}$. This means that $\mathcal{A}$ admits a layer (R, W) such that: R is a minimal k-algebra, the layer (R, W) is triangular, the R-R-bimodule W_1 is freely generated by a finite directed subset $\mathbb{B}_1$ of W_1, and the bimodule filtration

$$W_0^0 \subseteq W_0^1 \subseteq \ldots \subseteq W_0^{\ell_0} = W_0$$

corresponding to W_0 in the triangularity conditions for the layer, see [4, 5.1], is freely generated by a set filtration

$$\mathbb{B}_0^0 \subseteq \mathbb{B}_0^1 \subseteq \ldots \subseteq \mathbb{B}_0^{\ell_0} = \mathbb{B}_0$$

of a finite directed subset $\mathbb{B}_0$ of W_0. This means that each W_0^i is freely generated by $\mathbb{B}_0^i$, as in [4, 23.2].

Recall that a *rational algebra* Γ is, by definition, a finitely generated localization of the polynomial algebra $k[x]$. By definition, a minimal algebra R is a finite product of the form $k \times \cdots \times k \times \Gamma_1 \times \cdots \times \Gamma_t$, where $\Gamma_1, \ldots, \Gamma_t$ are rational algebras.

There is a bigraph $\mathbb{B}$ attached naturally to any seminested ditalgebra $\mathcal{A}$, see [4, 23.9]. The *points* in $\mathbb{B}$ are in bijective correspondence with the indecomposable factors of R, and the *marked points* are by definition those corresponding to factors which are rational algebras. The sets $\mathbb{B}_0$ and $\mathbb{B}_1$ are, respectively, the sets of *solid arrows* and *dashed arrows* of the bigraph $\mathbb{B}$ (and of the seminested ditalgebra $\mathcal{A}$). The bigraph $\mathbb{B}$ of $\mathcal{A}$ allows us to describe the category $\mathcal{A}$-Mod as a category of representations of the bigraph, see [4, 23.10].

Finally, we recall that a ditalgebra $\mathcal{A}$ over an algebraically closed field k is *tame* if, for every $d \in \mathbb{N}$, there is a finite collection $\{(\Gamma_i, Z_i)\}_{i=1}^n$, where Γ_i is a rational algebra and Z_i is an A-Γ_i-bimodule which is free of finite rank as a right Γ_i-module, such that, for every indecomposable $M \in \mathcal{A}$-Mod with $\dim_k M = d$, there are an $i \in [1, n]$ and a simple Γ_i-module S with $Z_i \otimes_{\Gamma_i} S \cong M$ in $\mathcal{A}$-Mod. There are various reformulations of this definition, see [4, § 27].

In the following, we adapt to the context of tame seminested ditalgebras some definitions and results on tame finite-dimensional algebras due to Crawley-Boevey, see [5, § 5]. Some of these adaptations are derived directly from his results (this is the case of Proposition 2.11); some others use his arguments rephrased for ditalgebras in [4]. Given a rational algebra Γ, we denote by $\mathrm{Irred}\,\Gamma$ a complete set of inequivalent irreducible elements of Γ.

Definition 2.4.

Let $\mathcal{A}$ be a tame seminested ditalgebra over the field k, as in [4, 23.5]. If G is a pregeneric $\mathcal{A}$-module, a *realization Z for G* over the rational algebra $\Gamma = k[x]_f$ is an A-Γ-bimodule Z, finitely generated as a right Γ-module, such that

$$G \cong Z \otimes_\Gamma k(x) \quad \text{in} \quad \mathcal{A}\text{-Mod} \qquad \text{and} \qquad \mathrm{endol}\, G = \dim_{k(x)}(Z \otimes_\Gamma k(x)).$$

Remark 2.5.

If $\mathcal{A}$ is a layered k-ditalgebra and $G \cong Z \otimes_\Gamma k(x)$ in $\mathcal{A}$-Mod, for some A-Γ-bimodule Z, where Γ is a rational algebra, then we have the canonical embeddings of k-algebras

$$k(x) \subseteq \mathrm{End}_{A^{k(x)}}(Z \otimes_\Gamma k(x)) \subseteq \mathrm{End}_A(Z \otimes_\Gamma k(x)) \subseteq \mathrm{End}_{\mathcal{A}}(Z \otimes_\Gamma k(x)),$$

where $A^{k(x)}$ denotes the extended algebra $A \otimes_k k(x)$, and hence $\mathrm{endol}\, G = \mathrm{endol}\,(Z \otimes_\Gamma k(x)) \leq \dim_{k(x)}(Z \otimes_\Gamma k(x))$.

Theorem 2.6.

Let $\mathcal{A}$ be a tame seminested ditalgebra and $d \in \mathbb{N}$. Then, there are a minimal ditalgebra $\mathcal{B}$, see [4, 23.5], and an A-B-bimodule Y_d, which is finitely generated as a right B-module, such that, for any $G \in \mathcal{A}$-Mod with $\mathrm{endol}\, G \leq d$, there are a B-E-bimodule N with finite length as a right E-module and an isomorphism $G \cong Y_d \otimes_B N$ in $\mathcal{A}$-Mod, where $E = \mathrm{End}_{\mathcal{A}}(G)^{\mathrm{op}}$. Moreover, there is a full and faithful functor $F\colon \mathcal{B}\text{-Mod} \to \mathcal{A}\text{-Mod}$ such that the following diagram commutes up to isomorphism:

$$\begin{array}{ccc} B\text{-Mod} & \xrightarrow{L_{\mathcal{B}}} & \mathcal{B}\text{-Mod} \\ {\scriptstyle Y_d \otimes_B -}\downarrow & & \downarrow{\scriptstyle F} \\ A\text{-Mod} & \xrightarrow{L_{\mathcal{A}}} & \mathcal{A}\text{-Mod}, \end{array}$$

where $L_{\mathcal{A}}$ and $L_{\mathcal{B}}$ denote the canonical embeddings.

Proof. Apply [4, 28.22] to any given d, to obtain a minimal ditalgebra $\mathcal{B}$ and a reduction functor $F\colon \mathcal{B}\text{-Mod} \to \mathcal{A}\text{-Mod}$ such that, for any k-algebra E, the induced functor $F^E\colon \mathcal{B}\text{-}E\text{-Mod} \to \mathcal{A}\text{-}E\text{-Mod}$ is length controlling and, for any $\mathcal{A}$-E-bimodule G with length $\leq d$, there is a $\mathcal{B}$-E-bimodule N such that $G \cong F^E(N)$. By definition, a *reduction functor*, see [4, 25.10], is a composition of functors of type F^a, F^r, F^d, F^e and F^u, corresponding to ditalgebra operations of type: absorption of a loop, as in [4, 23.16], regularization, as in [4, 23.15], deletion of idempotents, as in [4, 23.14], edge reduction, as in [4, 23.18], and unravelling, as in [4, 23.23], respectively. The functors F^a, F^r and F^d are full and faithful, by [4, 8.20], [4, 8.19] and [4, 8.17], respectively. The functors F^e and F^u are full and faithful because they are of type F^X, where X is a complete admissible module, by [4, 17.12]. It follows that any reduction functor is full and faithful, and so is F. From [4, 22.7], we get that $Y_d = F(B)$ has the structure of an A-B-bimodule, finitely generated as a right B-module, and the above diagram commutes.

Assume that G is an $\mathcal{A}$-module with endolength $\leq d$. If we make $E = \mathrm{End}_{\mathcal{A}}(G)^{\mathrm{op}}$, then G is an $\mathcal{A}$-E-bimodule with length $\leq d$ as a right E-module, and has the form $G \cong F^E(N)$ for some $N \in \mathcal{B}\text{-}E\text{-Mod}$ with finite length. □

Proposition 2.7.

Let $\mathcal{A}$ be a tame seminested ditalgebra over the algebraically closed field k and take $d \in \mathbb{N}$. Then, if $\mathcal{B}$ and Y_d are the minimal k-ditalgebra and the A-B-bimodule obtained by applying Theorem 2.6, with the integer d, we have:

(i) *The $\mathcal{A}$-modules of the form $G = Y_d \otimes_B Q_z$, where Q_z is some principal generic B-module, see [4, 31.3(1)], are pregeneric, and satisfy*

$$\mathrm{End}_{\mathcal{A}}\, G \big/ \mathrm{rad}\, \mathrm{End}_{\mathcal{A}}\, G \cong k(x).$$

(ii) *Any pregeneric $\mathcal{A}$-module of endolength $\leq d$ arises this way.*

Proof. (i) For any marked point z of the minimal ditalgebra $\mathcal{B}$, denote by Q_z the principal generic B-module at the point z. Thus, $Be_z = k[x]_{f(x)}$ and $Q_z = k(x)$ has a natural structure of a B-$k(x)$-bimodule. Consider the reduction functor $F\colon \mathcal{B}\text{-Mod} \to \mathcal{A}\text{-Mod}$ of the last theorem, then $G = Y_d \otimes_B Q_z \cong F(Q_z)$ is an $\mathcal{A}^{k(x)}$-module, finite-dimensional over $k(x)$. Thus, from Remark 2.5, the $\mathcal{A}$-module G has finite endolength. Proceeding as in the proof of [4, 31.7], we obtain (i).

(ii) Let G be a pregeneric $\mathcal{A}$-module with $\mathrm{endol}\, G \leq d$. By assumption, we already know that $G \cong Y_d \otimes_B N$, for some B-E-bimodule N, where $E = \mathrm{End}_{\mathcal{A}}(G)^{\mathrm{op}}$. Then, N is a generic B-module and so $N \cong Q_z$, for some marked point z of $\mathcal{B}$, by [4, 31.3]. □

Corollary 2.8.

Assume that $\mathcal{A}$ is a seminested ditalgebra over the algebraically closed field k. Then, $\mathcal{A}$ is tame if and only if it is pregenerically tame.

Proof. From Drozd's theorem, $\mathcal{A}$ is tame if and only if $\mathcal{A}$ is not wild, see [6] and [4, 27.10]. Then, [3, 2.9] gives that $\mathcal{A}$ is tame whenever it is pregenerically tame. Finally, by Proposition 2.7, the tameness of $\mathcal{A}$ implies its pregeneric tameness. □

Remark 2.9.

Let $\mathcal{A}$ be a tame seminested k-ditalgebra and $G \cong Z \otimes_\Gamma k(x)$ in $\mathcal{A}$-Mod, as in Remark 2.5. Then, from Proposition 2.7, it follows that

$$\operatorname{endol} G = \dim_{k(x)}(Z \otimes_\Gamma k(x)).$$

In particular, the last equality in the definition of realization can be eliminated.

Theorem 2.10.

Let $\mathcal{A}$ be a tame seminested ditalgebra. Then:

(i) *For any pregeneric $\mathcal{A}$-module G, there is a realization Z of G, over some rational algebra Γ, which is free as a right Γ-module and such that the composition*

$$\Gamma\text{-Mod} \xrightarrow{Z\otimes_\Gamma -} A\text{-Mod} \xrightarrow{L_{\mathcal{A}}} \mathcal{A}\text{-Mod}$$

preserves indecomposables and isomorphism classes.

(ii) *For each $d \in \mathbb{N}$, there are pregeneric $\mathcal{A}$-modules $G_1, \dots, G_m$ and, for each $i \in [1, m]$, a realization Z_i of G_i over a rational algebra Γ_i, such that, for almost all indecomposable $\mathcal{A}$-modules M with $\dim_k M \leq d$, we have an isomorphism $M \cong Z_i \otimes_{\Gamma_i} \Gamma_i/(p^n)$ in $\mathcal{A}$-Mod for some $i \in [1, m]$, $p \in \operatorname{Irred} \Gamma_i$, and $n \in \mathbb{N}$.*

Proof. (i) If G is a pregeneric $\mathcal{A}$-module with endolength d, then applying Theorem 2.6, we obtain a minimal ditalgebra $\mathcal{B}$, a reduction functor $F\colon \mathcal{B}\text{-Mod} \to \mathcal{A}\text{-Mod}$, and an A-B-bimodule Z such that $L_{\mathcal{A}}(Z \otimes_B -) \cong F L_{\mathcal{B}}$. From the last proposition, we know that $G \cong F(Q_z)$, for some principal generic B-module Q_z. Consider the rational algebra $\Gamma_z = Be_z = k[x]_f$. Then, from [4, 22.7], we have the following diagram, which commutes up to isomorphism:

$$\begin{array}{ccccc}
\Gamma_z\text{-Mod} & \xrightarrow{\Gamma_z\otimes_{\Gamma_z}-} & B\text{-Mod} & \xrightarrow{L_{\mathcal{B}}} & \mathcal{B}\text{-Mod} \\
\| & & \downarrow{\scriptstyle \underline{F}} & & \downarrow{\scriptstyle F} \\
\Gamma_z\text{-Mod} & \xrightarrow{F(\Gamma_z)\otimes_{\Gamma_z}-} & A\text{-Mod} & \xrightarrow{L_{\mathcal{A}}} & \mathcal{A}\text{-Mod}.
\end{array}$$

From [4, 31.6], $L_{\mathcal{B}}$ preserves indecomposables and isomorphism classes. Hence, the composition $F L_{\mathcal{B}}(\Gamma_z \otimes_{\Gamma_z} -)$ preserves indecomposability and isomorphism classes and, therefore, so does the lower row of the diagram $L_{\mathcal{A}}(F(\Gamma_z) \otimes_{\Gamma_z} -)$.

Moreover, since F is a reduction functor, $F(\Gamma_z)$ is an A-Γ_z-bimodule which is projective and finitely generated by the right. Hence, since Γ_z is a principal ideal domain, $F(\Gamma_z) \cong Z \otimes_B \Gamma_z \cong Z e_z$ is in fact a free right Γ_z-module of finite rank. We have in $\mathcal{A}$-Mod the isomorphisms

$$F(\Gamma_z) \otimes_{\Gamma_z} k(x) = L_{\mathcal{A}}\big(F(\Gamma_z) \otimes_{\Gamma_z} Q_z\big) \cong F\big(L_{\mathcal{B}}(\Gamma_z \otimes_{\Gamma_z} Q_z)\big) \cong F(Q_z) \cong G$$

and $\dim_{k(x)}(F(\Gamma_z) \otimes_{\Gamma_z} k(x)) = \operatorname{rk} F(\Gamma_z) = \operatorname{endol} G$, see the proof of [4, 31.8]. Thus, the bimodule $F(\Gamma_z)$ is the wanted realization for G over Γ_z.

(ii) From [4, 29.6], the reduction functor which appeared in Theorem 2.6 also satisfies that, for any indecomposable $\mathcal{A}$-module M with $\dim_k M \leq d$, there is a $\mathcal{B}$-module N with $F(N) \cong M$. Moreover, almost any such finite-dimensional indecomposable $\mathcal{B}$-module N is of the form $N \cong \Gamma_z/(p^i)$, for some marked point z of $\mathcal{B}$, some $p \in \operatorname{Irred}\Gamma_z$, and $i \in \mathbb{N}$. Thus, with the notation of the last argument, for almost any indecomposable $\mathcal{A}$-module M with $\dim_k M \leq d$, there is such an $N \cong \Gamma_z/(p^i)$, and $M \cong Ze_z \otimes N \cong Ze_z \otimes \Gamma_z/(p^i)$, where Ze_z is a realization over the rational algebra Γ_z of the pregeneric $\mathcal{A}$-module $G_z = Z \otimes_B Q_z$. □

Proposition 2.11.

Let Λ be a tame finite-dimensional basic algebra over the algebraically closed field k and consider its Drozd's ditalgebra $\mathcal{D}$, as in [4, 19.1]. Let Z_1 and Z_2 be realizations of the pregeneric $\mathcal{D}$-modules G_1 and G_2, over the rational algebras Γ_1 and Γ_2, respectively. If there is an infinite subset P of $\operatorname{Irred}\Gamma_2$ such that, for all $p \in P$, we have

$$Z_2 \otimes_{\Gamma_2} \Gamma_2/(p^{i_p}) \cong Z_1 \otimes_{\Gamma_1} \Gamma_1/(q_p) \qquad \text{in} \quad \mathcal{D}\text{-Mod},$$

for some $q_p \in \Gamma_1$ and $i_p \in \mathbb{N}$, then $G_2 \cong G_1$.

Proof. Consider the composable functors $\mathcal{D}\text{-Mod} \xrightarrow{\Xi_\Lambda} \mathcal{P}^1(\Lambda) \xrightarrow{\text{Cok}} \Lambda\text{-Mod}$, where Ξ_Λ is the usual equivalence functor of [4, 19.8] and Cok is the cokernel functor, see [4, 18.10]. From [4, 22.18 (2)], if Z is the transition bimodule, we have

$$Z \otimes_D Z_2 \otimes_{\Gamma_2} \Gamma_2/(p^{i_p}) \cong \operatorname{Cok} \Xi_\Lambda\big[Z_2 \otimes_{\Gamma_2} \Gamma_2/(p^{i_p})\big] \cong \operatorname{Cok} \Xi_\Lambda\big[Z_1 \otimes_{\Gamma_1} \Gamma_1/(q_p)\big] \cong Z \otimes_D Z_1 \otimes_{\Gamma_1} \Gamma_1/(q_p).$$

Moreover, for $i \in [1,2]$, the relation $G_i \cong Z_i \otimes_{\Gamma_i} k(x)$ implies that

$$\operatorname{Cok} \Xi_\Lambda(G_i) \cong \operatorname{Cok} \Xi_\Lambda[Z_i \otimes_{\Gamma_i} k(x)] \cong Z \otimes_D Z_i \otimes_{\Gamma_i} k(x),$$

where the last term is finite-dimensional over $k(x)$. Thus, $\operatorname{Cok} \Xi_\Lambda(G_i)$ is a generic Λ-module with realization $Z \otimes_D Z_i$ over the rational algebra Γ_i. From [5, 5.2 (4)], we obtain that $\operatorname{Cok} \Xi_\Lambda(G_1) \cong \operatorname{Cok} \Xi_\Lambda(G_2)$. Hence, $G_1 \cong G_2$. □

3. Pregeneric modules for Drozd's ditalgebras

The proof of our main result relies on the following theorem, proved in [2]. It applies to tame seminested ditalgebras with a proper subditalgebra. Let us recall some terminology from [4].

Definition 3.1.

Let $\mathcal{A} = (T, \delta)$ be any ditalgebra with layer (R, W). Assume we have R-R-bimodule decompositions $W_0 = W_0' \oplus W_0''$ and $W_1 = W_1' \oplus W_1''$. Consider the subalgebra T' of T generated by R and $W' = W_0' \oplus W_1'$. Then, $A' = [T']_0$ is freely generated by the pair (R, W_0'). Let us also assume that $\delta(W_0') \subseteq A'W_1'A'$ and $\delta(W_1') \subseteq A'W_1'A'W_1'A'$. Then, the differential δ on T restricts to a differential δ' on the algebra T', and we obtain a new ditalgebra $\mathcal{A}' = (T', \delta')$ with layer (R, W'). A layered ditalgebra $\mathcal{A}'$ is called a *proper subditalgebra of* $\mathcal{A}$ if it is obtained from an R-R-bimodule decomposition of W, as we have just described.

A proper subditalgebra $\mathcal{A}'$ of a triangular ditalgebra $\mathcal{A}$ is called *initial* when its triangular filtrations coincide with the first terms of the triangular filtrations of $\mathcal{A}$, see [4, 14.8]. The inclusion $r\colon T' \to T$ yields a morphism of ditalgebras $r\colon \mathcal{A}' \to \mathcal{A}$ and, hence, a *restriction functor*

$$R^{\mathcal{A}}_{\mathcal{A}'} = F_r\colon \ \mathcal{A}\text{-Mod} \to \mathcal{A}'\text{-Mod}.$$

The projection $\pi\colon A = [T]_0 \to [T']_0 = A'$ yields an *extension functor*

$$E^{A}_{A'} = F_\pi\colon \ A'\text{-Mod} \to A\text{-Mod}.$$

Theorem 3.2.

Assume that $\mathcal{A}'$ is an initial subditalgebra of the tame seminested ditalgebra $\mathcal{A}$, over the algebraically closed field k. Then, for any $d \in \mathbb{N}$, there is a finite family $\mathcal{I}(d)$ of indecomposable $\mathcal{A}'$-modules such that, for any indecomposable $\mathcal{A}$-module M with $\dim_k M \leq d$ and $M \ncong E^{\mathcal{A}}_{\mathcal{A}'}(N)$ in $\mathcal{A}$-Mod, for any $N \in \mathcal{A}'$-Mod, the module $R^{\mathcal{A}}_{\mathcal{A}'}(M)$ is isomorphic in $\mathcal{A}'$-Mod to a direct sum of modules in $\mathcal{I}(d)$.

The following lemma is another important ingredient of the proof of our main result.

Lemma 3.3.

Let $\mathcal{A}$ be a Roiter ditalgebra with layer (R, W), where W_1 is a finitely generated R-R-bimodule, over an algebraically closed field, see [4, 5.5]. Assume that $\mathcal{A}^X$ is obtained from $\mathcal{A}$ by reduction, using the $\mathcal{A}'$-module X, where $\mathcal{A}'$ is an initial subditalgebra of $\mathcal{A}$ and X is a finite direct sum of pairwise non-isomorphic finite-dimensional indecomposable $\mathcal{A}'$-modules, see [4, 12.7–12.9]. Then:

(i) *The algebra $\Gamma = \mathrm{End}_{\mathcal{A}'}(X)^{\mathrm{op}}$ admits the splitting $\Gamma = S \oplus P$, where P is the radical of Γ, and $\mathcal{A}^X$ is a ditalgebra with triangular layer (S, W^X).*

(ii) *Let $F^X : \mathcal{A}^X\text{-Mod} \to \mathcal{A}\text{-Mod}$ be the associated functor, as in [4, 12.10]. Then, the $\mathcal{A}$-modules M of the form $M \cong F^X(N)$, for some (resp. finite-dimensional) $N \in \mathcal{A}^X$-Mod, are precisely the $\mathcal{A}$-modules such that its restriction $R^{\mathcal{A}}_{\mathcal{A}'}(M)$ is isomorphic in $\mathcal{A}'$-Mod to a (resp. finite) direct sum of direct summands of X.*

Proof. (i) We know that $\mathcal{A}$ is a Roiter ditalgebra and, by [4, 12.3], so is $\mathcal{A}'$. Therefore, since k is algebraically closed, from [4, 17.3], the $\mathcal{A}'$-module X is indeed admissible. Thus, $\mathcal{A}^X$ and F^X are defined. The module X is triangular, as in [4, 14.6], because S is semisimple. Hence, from [4, 14.10], the ditalgebra $\mathcal{A}^X$ has triangular layer (S, W^X).

(ii) This follows from [4, 25.5]. See the argument in the proof of [3, 7.3 (2)]. □

Theorem 3.4.

Let Λ be a tame finite-dimensional basic algebra over the algebraically closed field k and consider its Drozd's ditalgebra $\mathcal{D}$. Assume that $\mathcal{D}'$ is an initial subditalgebra of the tame seminested ditalgebra $\mathcal{D}$ and that $E^{\mathcal{D}}_{\mathcal{D}'}(M) \cong E^{\mathcal{D}}_{\mathcal{D}'}(N)$ in $\mathcal{D}$-Mod whenever $M \cong N$ in $\mathcal{D}'$-Mod. Then, for any $d \in \mathbb{N}$, there is a finite family $\mathcal{I}(d)$ of finite-dimensional indecomposable $\mathcal{D}'$-modules such that

(i) *for any indecomposable $\mathcal{D}$-module M with $\dim_k M \leq d$ and $M \ncong E^{\mathcal{D}}_{\mathcal{D}'}(N)$ in $\mathcal{D}$-Mod, for any $\mathcal{D}'$-module N, the module $R^{\mathcal{D}}_{\mathcal{D}'}(M)$ is isomorphic in $\mathcal{D}'$-Mod to a direct sum of modules of $\mathcal{I}(d)$;*

(ii) *for any pregeneric $\mathcal{D}$-module G with $\mathrm{endol}\, G \leq d$ and $G \ncong E^{\mathcal{D}}_{\mathcal{D}'}(H)$ in $\mathcal{D}$-Mod, for any pregeneric $\mathcal{D}'$-module H, the module $R^{\mathcal{D}}_{\mathcal{D}'}(G)$ is isomorphic in $\mathcal{D}'$-Mod to a direct sum of modules of $\mathcal{I}(d)$.*

Proof. From [4, 22.13], we know that $\mathcal{D}'$ is also tame. Fix $d \in \mathbb{N}$ and apply Theorem 3.2 to $\mathcal{D}$ and $\mathcal{D}'$, to obtain a finite set $\mathcal{I}(d) = \{X_1, \ldots, X_t\}$ of pairwise non-isomorphic finite-dimensional indecomposable $\mathcal{D}'$-modules satisfying (i). Let G be a pregeneric $\mathcal{D}$-module such that $\mathrm{endol}\, G \leq d$ and $G \ncong E(H)$, for any pregeneric $\mathcal{D}'$-module H.

From (i) of Theorem 2.10 there is a realization Z of G over a rational algebra Γ, which is free finitely generated as a right Γ-module. It defines the infinite family of pairwise non-isomorphic indecomposable $\mathcal{D}$-modules

$$\{Z \otimes_\Gamma \Gamma/(p) : p \in \mathrm{Irred}\, \Gamma\}.$$

If $\mathrm{rk}\, Z$ denotes the rank of Z as a free right Γ-module, then $\mathrm{rk}\, Z = \dim_{k(x)}(Z \otimes_\Gamma k(x)) = \mathrm{endol}\, G \leq d$ and, for each $p \in \mathrm{Irred}\, \Gamma$, we have that

$$\dim_k(Z \otimes_\Gamma \Gamma/(p)) \leq d.$$

Then, for any $p \in \mathrm{Irred}\, \Gamma$ with $Z \otimes_\Gamma \Gamma/(p) \ncong E^{\mathcal{D}}_{\mathcal{D}'}(N)$ in $\mathcal{D}$-Mod, for any $N \in \mathcal{D}'$-Mod, the module $R^{\mathcal{D}}_{\mathcal{D}'}(Z \otimes_\Gamma \Gamma/(p))$ is isomorphic in $\mathcal{D}'$-Mod to a direct sum of modules in $\mathcal{I}(d)$. Consider the admissible $\mathcal{D}'$-module $X = \bigoplus_{i=1}^t X_i$, the seminested ditalgebra $\mathcal{D}^X$, see [3, 3.4], and the associated full and faithful reduction functor $F^X : \mathcal{D}^X\text{-Mod} \to \mathcal{D}\text{-Mod}$. Let us first show the following.

Claim. *There is no infinite subset P of* $\operatorname{Irred}\Gamma$ *such that, for all $p \in P$, there is $N_p \in \mathcal{D}'$-Mod with $Z \otimes_\Gamma \Gamma/(p) \cong E(N_p)$.*

Proof of the claim. Assume that there is such a set P. Then, the tame seminested ditalgebra $\mathcal{D}'$ admits an infinite family $\{N_p\}_{p\in P}$ of pairwise non-isomorphic indecomposable $\mathcal{D}'$-modules with $\dim_k N_p \leq d$. Then, from (ii) of Theorem 2.10 there are a pregeneric $\mathcal{D}'$-module G', a realization Z' of G', over some rational algebra Γ', and an infinite subset Q of $\operatorname{Irred}\Gamma'$ such that, for any $q \in Q$, there are $p_q \in P$ and $i_q \in \mathbb{N}$ with $Z' \otimes_{\Gamma'} \Gamma'/(q^{i_q}) \cong N_{p_q}$. Then, for all $q \in Q$, we have

$$Z \otimes_\Gamma \Gamma/(p_q) \cong E(N_{p_q}) \cong E(Z' \otimes_{\Gamma'} \Gamma'/(q^{i_q})) \cong E(Z') \otimes_{\Gamma'} \Gamma'/(q^{i_q}).$$

Moreover, $E(G') \cong E(Z' \otimes_{\Gamma'} k(x)) \cong E(Z') \otimes_{\Gamma'} k(x)$. From Remark 2.9, we have that $E(G')$ is a pregeneric $\mathcal{D}$-module with realization $E(Z')$ over Γ'. Then, from Proposition 2.11, we obtain that $E(G') \cong G$, contradicting our initial assumption. This ends the proof of our claim. ■

Then, there are infinitely many elements $p \in \operatorname{Irred}\Gamma$ such that

$$Z \otimes_\Gamma \Gamma/(p) \ncong E(N) \qquad \text{for any} \quad N \in \mathcal{D}'\text{-Mod}.$$

Hence, there is an infinite subset $P \subseteq \operatorname{Irred}\Gamma$ such that, for any $p \in P$, the module $R^{\mathcal{D}}_{\mathcal{D}'}(Z \otimes_\Gamma \Gamma/(p))$ is isomorphic in $\mathcal{D}'$-Mod to a direct sum of direct summands of X. From Lemma 3.3, we know that, for each $p \in P$, there is a $\mathcal{D}^X$-module L_p with $Z \otimes_\Gamma \Gamma/(p) \cong F^X(L_p)$.

The tame seminested ditalgebra $\mathcal{D}^X$ admits the infinite family $\{L_p\}_{p\in P}$ of pairwise non-isomorphic indecomposable $\mathcal{D}^X$-modules with bounded dimension. From (ii) of Theorem 2.10, there are a pregeneric $\mathcal{D}^X$-module G', a realization Z' of G', over some rational algebra Γ', and an infinite subset Q of $\operatorname{Irred}\Gamma'$ such that, for any $q \in Q$, there are $p_q \in P$ and $i_q \in \mathbb{N}$ with

$$Z' \otimes_{\Gamma'} \Gamma'/(q^{i_q}) \cong L_{p_q}.$$

Thus, for $q \in Q$, we have

$$Z \otimes_\Gamma \Gamma/(p_q) \cong F^X(L_{p_q}) \cong F^X(Z' \otimes_{\Gamma'} \Gamma'/(q^{i_q})) \cong F^X(Z') \otimes_{\Gamma'} \Gamma'/(q^{i_q}).$$

Moreover, $F^X(G') \cong F^X(Z' \otimes_{\Gamma'} k(x)) \cong F^X(Z') \otimes_{\Gamma'} k(x)$. From Remark 2.9, we obtain that $F^X(G')$ is a pregeneric $\mathcal{D}$-module and $F^X(Z')$ is a realization of $F^X(G')$ over Γ'. Then, from Proposition 2.11, we obtain that $F^X(G') \cong G$. Hence, from Lemma 3.3, the module $R^{\mathcal{D}}_{\mathcal{D}'}(G)$ is a direct sum of direct summands of X in $\mathcal{D}'$-Mod. □

4. Main result for algebras

The first statement of the following theorem was proved in [2]. The proof of the fact that the same set $\mathcal{I}_0(d)$ works for the second statement is somehow parallel to the proof given in [2, 4.1]. For the benefit of the reader, we provide a complete proof, after recalling some constructions from [2].

Reminder 4.1.

Let $\mathcal{D} = (T, \delta)$ be a seminested ditalgebra with layer (R, W) and set of points $\mathcal{P}$. Assume that $\mathcal{D}' = (T', \delta')$ is a proper subditalgebra of $\mathcal{D}$ associated to the R-R-bimodule decompositions $W_0 = W_0' \oplus W_0''$ and $W_1 = W_1' \oplus W_1''$. Then, the subditalgebra $\mathcal{D}'$ is called *convex* if there is a subset $\mathcal{P}_0$ of $\mathcal{P}$ such that $eW_0'e = W_0'$ and $eW_1'e = W_1'$, where e is the central idempotent $e = \sum_{x\in\mathcal{P}_0} e_x$ of R. It follows that $\mathcal{D}'$ is seminested.

If $\mathcal{D}'$ is a convex subditalgebra of the seminested ditalgebra $\mathcal{D}$, the morphism of algebras $\eta\colon T \to T'$ determined by the projection of R-R-bimodules $W \to W'$ is a morphism of ditalgebras $\eta\colon \mathcal{D} \to \mathcal{D}'$, which induces a restriction functor $F_\eta\colon \mathcal{D}'\text{-Mod} \to \mathcal{D}\text{-Mod}$ with $F_\eta(M) = E^{D}_{D'}(M)$, for $M \in \mathcal{D}'$-Mod. Thus, $E^{D}_{D'}(M) \cong E^{D}_{D'}(N)$ in $\mathcal{D}$-Mod whenever $M \cong N$ in $\mathcal{D}'$-Mod.

Moreover, if we write $f = 1 - e$, we get $R = Re \times Rf$ and an isomorphism of ditalgebras $\mathcal{D}' \cong \mathcal{D}^e \times \mathcal{D}^f$, where $\mathcal{D}^e$ and $\mathcal{D}^f$ are ditalgebras with layers (Re, W') and $(Rf, 0)$, see [2, 5.2]. In particular, the category $\mathcal{D}^f$-Mod can be identified with Rf-Mod.

Since $\mathcal{D}' \cong \mathcal{D}^e \times \mathcal{D}^f$, we can consider the projection morphisms $\pi^e\colon \mathcal{D}' \to \mathcal{D}^e$ and $\pi^f\colon \mathcal{D}' \to \mathcal{D}^f$. The induced functors $F^e\colon \mathcal{D}^e\text{-Mod} \to \mathcal{D}'\text{-Mod}$ and $F^f\colon \mathcal{D}^f\text{-Mod} \to \mathcal{D}'\text{-Mod}$ determine an equivalence of categories

$$\mathcal{D}^e\text{-Mod} \times \mathcal{D}^f\text{-Mod} \xrightarrow{F^e \oplus F^f} \mathcal{D}'\text{-Mod}$$

described in [4, 10.3].

Now, assume that Λ is a basic finite-dimensional algebra over the algebraically closed field k and Λ_0 is a convex algebra in Λ. Then, there are a convex subditalgebra $\mathcal{D}'$ of the Drozd ditalgebra $\mathcal{D}$ of Λ and a functor $\Xi'\colon \mathcal{D}'\text{-Mod} \to \mathcal{P}^1(\Lambda_0)$ such that the following diagram commutes up to isomorphism:

$$\begin{array}{ccccc} \mathcal{D}\text{-Mod} & \xrightarrow{\Xi_\Lambda} & \mathcal{P}^1(\Lambda) & \xrightarrow{\text{Cok}} & \Lambda\text{-Mod} \\ \downarrow{\scriptstyle R^{\mathcal{D}}_{\mathcal{D}'}} & & \downarrow{\scriptstyle \text{Res}} & & \downarrow{\scriptstyle \text{res}} \\ \mathcal{D}'\text{-Mod} & \xrightarrow{\Xi'} & \mathcal{P}^1(\Lambda_0) & \xrightarrow{\text{Cok}_0} & \Lambda_0\text{-Mod}, \end{array}$$

where Ξ_Λ is the usual equivalence, see [4, 19.8], and Res is the restricted lifting of res, see [2, 2.1 and 5.3]. The functor Ξ' is constructed as the composition

$$\mathcal{D}'\text{-Mod} \xrightarrow{H} \mathcal{D}^e\text{-Mod} \xrightarrow{F_\varphi} \mathcal{D}^{\Lambda_0}\text{-Mod} \xrightarrow{\Xi_{\Lambda_0}} \mathcal{P}^1(\Lambda_0),$$

where H is the projection, F_φ is the functor induced by an isomorphism of seminested ditalgebras $\varphi\colon \mathcal{D}^{\Lambda_0} \to \mathcal{D}^e$, and Ξ_{Λ_0} is the usual equivalence.

Let us also recall that, given the convex subditalgebra $\mathcal{D}'$, we can modify the triangular filtrations of $\mathcal{D}$, obtaining a different seminested ditalgebra $\overline{\mathcal{D}}$ with the same underlying ditalgebra $\mathcal{D}$, such that $\mathcal{D}'$ is an initial convex subditalgebra of $\overline{\mathcal{D}}$. Thus, $\mathcal{D}$ and $\overline{\mathcal{D}}$ coincide as ditalgebras and share the same layer (but with different triangular filtrations). In particular, we have that $\overline{\mathcal{D}}\text{-Mod} = \mathcal{D}\text{-Mod}$ and $R^{\mathcal{D}}_{\mathcal{D}'} = R^{\overline{\mathcal{D}}}_{\mathcal{D}'}$. See [2, 5.4].

Theorem 4.2.

Assume that Λ is a tame finite-dimensional basic algebra over an algebraically closed field k. Suppose that Λ_0 is a convex algebra in Λ. Then, for any $d \in \mathbb{N}$, there is a finite family $\mathcal{I}_0(d)$ of finite-dimensional indecomposable Λ_0-modules such that

(i) *for any indecomposable Λ-module M with $\dim_k M \leq d$ and $M \ncong \operatorname{tens} N$, for any Λ_0-module N, the module $\operatorname{res} M$ is isomorphic to a direct sum of modules in $\mathcal{I}_0(d)$;*

(ii) *for any generic Λ-module G with $\operatorname{endol} G \leq d$ and $G \ncong \operatorname{tens} H$, for any generic Λ_0-module H, the module $\operatorname{res} G$ is isomorphic to a direct sum of modules in $\mathcal{I}_0(d)$.*

Proof. We adopt the notations introduced in Reminder 4.1. Thus, $\mathcal{D}$ is the Drozd ditalgebra associated to the algebra Λ. Since Λ is tame, from [4, 27.14], so are $\mathcal{D}$ and $\overline{\mathcal{D}}$ (recall that $\mathcal{D}\text{-Mod} = \overline{\mathcal{D}}\text{-Mod}$).

Fix $d \in \mathbb{N}$. Then, we apply Theorem 3.4 to $d' = (1 + \dim_k \Lambda) \times d$, to obtain a finite family $\mathcal{I}'(d')$ of finite-dimensional indecomposable $\mathcal{D}'$-modules such that, for any pregeneric $\overline{\mathcal{D}}$-module H with $\operatorname{endol} H \leq d'$ and $H \ncong E^{\overline{D}}_{D'}(H')$, for any pregeneric $H' \in \mathcal{D}'\text{-Mod}$, we have that $R^{\overline{\mathcal{D}}}_{\mathcal{D}'}(H)$ is isomorphic to a direct sum of indecomposables of $\mathcal{I}'(d')$. Notice that we really need to replace $\mathcal{D}$ by $\overline{\mathcal{D}}$, first, in Proposition 2.11 and, after, in Theorem 3.4, before we can derive the preceding statement for the tame seminested ditalgebras $\mathcal{D}'$ and $\overline{\mathcal{D}}$.

Since $\mathcal{D}'\text{-Mod}$ is equivalent to the product category $\mathcal{D}^e\text{-Mod} \times \mathcal{D}^f\text{-Mod}$, we can consider the subfamily $\mathcal{I}''(d')$ of $\mathcal{I}'(d')$ obtained from this last one by excluding all the indecomposables from $\mathcal{D}^f\text{-Mod}$, as well as all the indecomposables $N' \in \mathcal{D}^e\text{-Mod}$ such that $\Xi_{\Lambda_0} F_\varphi(N')$ has the form $Q \to 0$. Then, $\mathcal{I}(d) = \text{Cok}_0\, \Xi' \mathcal{I}''(d')$ is a finite family of finite-dimensional indecomposable Λ_0-modules.

Take any generic Λ-module M with $\operatorname{endol} M \leq d$ such that $M \ncong \operatorname{tens} H$, for any generic Λ_0-module H. Let us show that $\operatorname{res} M$ is isomorphic to a direct sum of Λ_0-modules in $\mathcal{I}(d)$. Consider a minimal projective presentation $Q' \to Q \to M \to 0$

of M. Then, there is an $N \in \mathcal{D}$-Mod $= \overline{\mathcal{D}}$-Mod such that $\Xi_\Lambda(N) \cong (Q' \to Q)$ and $\mathrm{Cok}\, \Xi_\Lambda(N) \cong M$. Since M is indecomposable, so is N. Then, from [3, 4.4], we obtain $\mathrm{endol}\, N \leq \mathrm{endol}\, M \times (1 + \dim_k \Lambda) \leq d'$.

Suppose that $N \cong E_{D'}^{\overline{D}}(N')$, for some pregeneric $N' \in \mathcal{D}'$-Mod. There is an isomorphism $N' \cong F^e(N^e) \oplus F^f(N^f)$ in D'-Mod, for some $N^e \in \mathcal{D}^e$-Mod and $N^f \in \mathcal{D}^f$-Mod, which is preserved by the functor $E_{D'}^{\overline{D}}$. Then, $N \cong E_{D'}^{\overline{D}}(N') \cong E_{D'}^{\overline{D}} F^e(N^e) \oplus E_{D'}^{\overline{D}} F^f(N^f)$ and, since N is indecomposable, we have that $N^e = 0$ or $N^f = 0$. If $N^f \neq 0$, we obtain $N^e = 0$ and N^f is indecomposable. In order to justify this last statement, assume N^f decomposes non-trivially, it does so in D^f-Mod, hence $F^f(N^f)$ has a non-trivial decomposition in D'-Mod, which is preserved by $E_{D'}^{\overline{D}}$, contradicting again the idecomposability of N. This argument is not superfluous, because the domain of $E_{D'}^{\overline{D}}$ is D'-Mod not $\mathcal{D}'$-Mod, thus we need to show that the decomposition of $F^f(N^f)$ occurs in fact in D'-Mod. Since $\mathcal{D}$ has no marked points, that is R is a product of copies of k, the $\mathcal{D}^f$-module N^f is one-dimensional. Thus, $F^f(N^f)$ is a one-dimensional module, corresponding to a point of $\mathcal{D}'$, not in $\mathcal{D}^e$. Then, its extension $N \cong E_{D'}^{\overline{D}} F^f(N^f)$ is again such a one-dimensional $\mathcal{D}$-module: a contradiction because N is infinite-dimensional. Then, we can assume that $N^f = 0$ and, hence, $N \cong E_{D'}^{\overline{D}} F^e(N^e)$.

As indicated in the proof of [2, 6.1], for any $N^e \in \mathcal{D}^e$-Mod, we have

$$\Xi_\Lambda E_{D'}^{\overline{D}} F^e(N^e) \cong \mathrm{Tens}\, \Xi_{\Lambda_0}(N^e),$$

where $\mathrm{Tens}\colon \mathcal{P}(\Lambda_0) \to \mathcal{P}(\Lambda)$ is the functor induced by tens on the categories of morphisms between projectives, see [2, 2.5]. Now, we apply this claim to our previously fixed N^e to obtain $\Xi_\Lambda(N) \cong \Xi_\Lambda E_{D'}^{\overline{D}} F^e(N^e) \cong \mathrm{Tens}\, \Xi_{\Lambda_0}(N^e)$. Therefore, using [2, 2.5], we get

$$M \cong \mathrm{Cok}\, \Xi_\Lambda(N) \cong \mathrm{Cok}\, \mathrm{Tens}\, \Xi_{\Lambda_0}(N^e) \cong \mathrm{tens}\, \mathrm{Cok}\, \Xi_{\Lambda_0}(N^e),$$

which leads to a contradiction: Indeed, N^e is a pregeneric $\mathcal{D}^e$-module because N' is a pregeneric $\mathcal{D}'$-module; thus, $\mathrm{Cok}\, \Xi_{\Lambda_0}(N^e)$ is a generic Λ_0-module.

Then, $N \ncong E_{D'}^{\overline{D}}(N')$, for any pregeneric $N' \in \mathcal{D}'$-Mod, and $R_{\mathcal{D}'}^{\overline{\mathcal{D}}}(N) \cong \bigoplus_i N_i$, for some indecomposable $\mathcal{D}'$-modules $N_i \in \mathcal{I}'(d')$. From the commutativity up to isomorphism of the diagram given in Reminder 4.1, it follows that

$$\mathrm{res}\, M \cong \mathrm{res}\, \mathrm{Cok}\, \Xi_\Lambda(N) \cong \mathrm{Cok}\, \mathrm{Res}\, \Xi_\Lambda(N) \cong \mathrm{Cok}\, \Xi' R_{\mathcal{D}'}^{\overline{\mathcal{D}}}(N) \cong \bigoplus_i \mathrm{Cok}\, \Xi'(N_i),$$

which is a direct sum of modules in $\mathcal{I}(d)$, and we are done. □

We immediately obtain the following.

Corollary 4.3.

Assume that Λ is a tame finite-dimensional basic algebra over an algebraically closed field k. Suppose that Λ_0 is a convex algebra in Λ. Then, for any generic Λ-module G, either the Λ_0-module $\mathrm{res}\, G$ is a generic Λ_0-module or $\mathrm{res}\, G$ is a direct sum of finite-dimensional indecomposable Λ_0-modules.

Remark 4.4.

Under the assumptions of the last corollary, for any Λ-module M, we have

$$\mathrm{endol}\, \mathrm{res}\, M \leq \dim_k \Lambda_0 \times (1 + \dim_k \Lambda) \times \mathrm{endol}\, M.$$

Indeed, assume that $\mathrm{endol}\, M = d$ is finite and keep in mind the notations of Reminder 4.1. Choose an indecomposable $N \in \overline{\mathcal{D}}$-Mod with $\mathrm{Cok}\, \Xi_\Lambda(N) \cong M$. Then, by [3, 2.2 and 4.4], we have $\mathrm{endol}\, R_{\mathcal{D}'}^{\overline{\mathcal{D}}}(N) \leq \mathrm{endol}\, N \leq (1 + \dim_k \Lambda) \times d$. As before, we have $\mathrm{res}\, M \cong \mathrm{Cok}_0\, \Xi' R_{\mathcal{D}'}^{\overline{\mathcal{D}}}(N)$ and then, using [3, 4.4], we obtain $\mathrm{endol}\, \mathrm{res}\, M \leq \dim_k \Lambda_0 \times \mathrm{endol}\, R_{\mathcal{D}'}^{\overline{\mathcal{D}}}(N)$.

Finally, we show an example of a wild finite-dimensional algebra Λ and a convex algebra Λ_0 in Λ for which the conclusions of Theorem 4.2 do not hold.

Example 4.5.
Consider the path k-algebra Λ of the quiver

$$1 \underset{\beta}{\overset{\alpha}{\leftleftarrows}} 2 \underset{\delta}{\overset{\gamma}{\rightrightarrows}} 3$$

and the convex algebra Λ_0 in Λ determined by the subquiver

$$2 \underset{\delta}{\overset{\gamma}{\rightrightarrows}} 3.$$

Then, we have the generic Λ-module G corresponding to the representation

$$k(x) \underset{\mathrm{Id}}{\overset{x}{\leftleftarrows}} k(x) \underset{\mathrm{Id}}{\overset{x}{\rightrightarrows}} k(x)$$

with $\mathrm{End}_\Lambda G \cong k(x)$ and $\mathrm{endol}\, G \leq 3$. The algebra Λ_0 is in fact cofinal in Λ, as in [2, 1.1]. Then, from [2, 2.3], the restriction $\mathrm{res} = \Lambda_0 \otimes_\Lambda -\colon \Lambda\text{-Mod} \to \Lambda_0\text{-Mod}$ is isomorphic to the standard restriction functor, which maps each $M \in \Lambda\text{-Mod}$ onto $(e_2+e_3)M$. Here, e_1, e_2, e_3 denote the canonical primitive orthogonal idempotents of Λ, corresponding to the vertices $1, 2, 3$, respectively, thus e_2+e_3 is the unit element in Λ_0. Hence, the Λ_0-module $G_0 = \mathrm{res}\, G$ is given by the representation

$$k(x) \underset{\mathrm{Id}}{\overset{x}{\rightrightarrows}} k(x)$$

and it is a generic Λ_0-module. Since Λ_0 is tame hereditary, we know that G_0 is the unique generic Λ_0-module, up to isomorphism, see [5, 1.5]. But $G \ncong \mathrm{tens}\, G_0$ because they are not isomorphic as right $k(x)$-modules. Indeed, the elements $\alpha \otimes e_2$ and $\beta \otimes e_2$ are $k(x)$-linearly independent, thus $\dim_{k(x)} e_1 \,\mathrm{tens}\, G_0 \geq 2 > \dim_{k(x)} e_1 G$. All this means that item (ii) of Theorem 4.2 does not hold here.

In order to see that item (i) of Theorem 4.2 fails too for Λ and Λ_0, we can consider a fixed $n \in \mathbb{N}$ and the family $\{M_\lambda\}_{\lambda \in k}$ of indecomposable Λ-modules with $\dim_k M_\lambda = 3n$ given by the representations

$$k^n \underset{I_n}{\overset{J_n(\lambda)}{\leftleftarrows}} k^n \underset{I_n}{\overset{J_n(\lambda)}{\rightrightarrows}} k^n,$$

where $J_n(\lambda)$ denotes the Jordan block with eigenvalue λ of size $n \times n$. For each $\lambda \in k$, the restriction $N_\lambda = \mathrm{res}\, M_\lambda$ is given by the representation

$$k^n \underset{I_n}{\overset{J_n(\lambda)}{\rightrightarrows}} k^n.$$

Then, they constitute an infinite family of pairwise non-isomorphic indecomposable Λ_0-modules. It is not hard to see that $M_\lambda \ncong \mathrm{tens}\, N_\mu$, for all $\lambda, \mu \in k$. Having in mind the well-known description of the indecomposable Λ_0-modules, see for instance [7, XI.4], and the fact that $\dim_k e_2 M_\lambda = n = \dim_k e_3 M_\lambda$, we can see that (i) of Theorem 4.2 does not hold.

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