International Journal of Algebra, Vol. 9, 2015, no. 2, 53 - 58
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/ija.2015.412123

On Almost Simple Transcendental Field Extensions

Antonio González Fernández, Rodrigo Jiménez Correa and Jesús Efrén Pérez Terrazas

Facultad de Matemáticas
Universidad Autónoma de Yucatán
Anillo Periférico Norte, 13615, Chuburná Hidalgo Inn
Mérida, Yucatán, México

Abstract

We study some properties of *almost simple transcendental field extensions* in order to analyze the endomorphisms ring of algebraically bounded $\Lambda-$modules where Λ is a semigenerically tame finite-dimensional $k-$algebra, k a perfect field.

Mathematics Subject Classification: 12E05, 16G60

Keywords: algebraically bounded modules, field extension, perfect field, semigenerically tame algebra, transcendental extension

1. Introduction

The notion of algebraically bounded module was introduced in [5] in order to study the representation type of finite dimensional algebras over perfect fields.

Here we deal with the ring of endomorphisms of such modules, and the main result is the following:

Corollary 1.1. *Let k be a perfect field, K an algebraic closure of k, and Λ a finite-dimensional $k-$algebra. Assume that Λ^K is tame and G is an algebraically bounded $\Lambda-$module. Let us denote $E_G = \mathrm{End}_\Lambda(G)^{op}$, $D_G = E_G/\mathrm{rad}(E_G)$, by Z_G the center of D_G and by A_G the algebraic elements of Z_G over k. Then A_G/k is a finite field extension and $Z_G = k(t, w)$, where t is transcendental over k and $k(t, w)/k(t)$ is a separable finite field extension. Also we have $E_G = F \oplus E'$ as $k-$vector spaces, F a subring of E_G and $F \cong k(t)$ as $k-$algebras, and G is finitely generated as right $F-$module.*

To achieve the previous result we study some properties of *almost simple transcendental* field extensions (see 3.1), through classical results of Field Theory.

In a forthcoming paper we will use the previous corollary and the results of [1] to produce bimodules that parametrize finite dimensional $\Lambda-$modules, when Λ is a semigenerically tame finite-dimensional $k-$algebra, with k a perfect field.

2. Basic Features

Lemma 2.1. *Let E/k be a field extension, $f \in k[x] - \{0\}$ and $g \in E[x]$ such that $fg \in k[x]$, then $g \in k[x]$.*

Proof: By the division algorithm there is an identity $fg = fq + r$ such that $q, r \in k[x]$ and $\mathrm{grad}(f) > \mathrm{grad}(r)$; then the identity $(g - q)f = r$ in $E[x]$ implies $r = 0$ and $g = q$. □

Proposition 2.2. *Let E/k be a field extension. Then the canonical homomorphism of $E-$algebras $\psi : E \otimes_k k(x) \cong E(x)$ is injective. If E/k is an algebraic field extension then ψ is surjective.*

Proof: It is easy to verify the existence of an homomorphism of $E-$algebras $\psi : E \otimes_k k(x) \to E(x)$ determined by $\psi\left(e \otimes \frac{f}{g}\right) = \frac{ef}{g}$, for $e \in E$, $f, g \in k[x]$, $g \neq 0$.

The restriction $\psi_{|E \otimes_k k[x]} : E \otimes_k k[x] \to E[x]$ is an isomorphism, and this implies that ψ is injective (use clearing denominators).

Now assume E/k is an algebraic field extension and let be $h \in E[x]$ with $\mathrm{grad}(h) = m > 0$, and E' a splitting field of h, i.e. $h = uh_1 \cdots h_m$ where $u \in E - \{0\}$ and $h_i = x - r_i$, $r_i \in E'$, for each i.

Let $h_i h_i' \in k[x]$ be the minimal polynomial of r_i over k, and so $hh' \in k[x]$, where $h' = u^{-1} h_1' \cdots h_m'$. By lemma 2.1 we get $h' \in E[x]$. Using the identity $\frac{1}{h} = \frac{h'}{hh'}$ we can verify that any element of $E(x)$ has the form $\sum_{i=0}^{n} \frac{e_i x^i}{g}$, where $e_i \in E$ for each i and $g \in k[x] - \{0\}$, and so ψ is surjective. □

Remark 2.3. $y \otimes 1 - 1 \otimes x$ is not a unit in $k(y) \otimes_k k(x)$.

Proposition 2.4. *Let E/k be an algebraic field extension, then $E(x)/k(x)$ is an algebraic field extension.*

Proof: Note that if $e \in E$ is a root of the polynomial $\sum\limits_{j=0}^{m} c_j x^j \in k[x]$ then $\frac{ex^i}{g}$ is a root of the polynomial $\sum\limits_{j=0}^{m} \left(c_j \frac{g^j}{x^{ij}}\right) y^j \in k(x)[y]$. From the proof of the proposition 2.2 we know that any element of $E(x)$ has the form $\sum\limits_{i=0}^{n} \frac{e_i x^i}{g}$, where $e_i \in E$ for each i and $g \in k[x] - \{0\}$. Then, the statement holds because the sum of algebraic elements is algebraic. □

Corollary 2.5. *Let E/k be an algebraic field extension. Then $E \otimes_k k(x_1, \ldots, x_n) \cong E(x_1, \ldots, x_n)$ as $E-$algebras.*

Proof: By induction over n. The inductive step follows by propositions 2.2, 2.4 and the identity $(k(x_1, \ldots, x_{m-1}))(x_m) = k(x_1, \ldots, x_m)$, and so we have

$$\begin{aligned} E \otimes_k k(x_1, \ldots, x_m) &\cong E \otimes_k k(x_1, \ldots, x_{m-1}) \otimes_{k(x_1,\ldots,x_{m-1})} k(x_1, \ldots, x_m) \\ &\cong E(x_1, \ldots, x_{m-1}) \otimes_{k(x_1,\ldots,x_{m-1})} (k(x_1, \ldots, x_{m-1}))(x_m) \\ &\cong E(x_1, \ldots, x_m). \end{aligned}$$

□

3. Properties of ast Field Extensions

Definition 3.1. Let E/k a field extension. We call E an *almost simple transcendental* field extension of k, or *ast*, if there exists an algebraic field extension L/k, and $n \in \mathbb{N}$, such that $E \otimes_k L \cong \bigtimes\limits_{i=1}^{n} L(x)$ as $L-$algebras. In this case we say that L realizes the ast property of E/k.

Applying proposition 2.2 we get the following result:

Lemma 3.2. *Assume that L realizes the ast property of E/k. If L'/L is an algebraic field extension then L' realizes the ast property of E/k.*

The next claim is theorem 7 of section 4 of chapter 4 of [4] (or see [2]).

Theorem 3.3. *Let t be transcendental over k. Then t is algebraic over $k(s)$ for $s \in k(t) - k$. Moreover, if $s = \frac{f(t)}{g(t)}$ with $f(x)$ and $g(x)$ relative primes in $k[x]$ then*

$$[k(t) : k(s)] = \max\{\deg(f), \deg(g)\}$$

and the polynomial $f(x) - sg(x)$ is irreducible in $k(s)[x]$.

Lemma 3.4. *Let α be algebraic over k and L/k a field extension, L containing a splitting field of $m_{\alpha,k}(x) \in k[x]$, the minimal (monic) polynomial of α over k.*

i) If α is not separable over k, then $L \otimes_k k(\alpha)$ contains nilpotent elements.

ii) If α is separable over k, then $L \otimes_k k(\alpha) \cong \overset{n}{\underset{i=1}{\times}} L$ as L-algebras, where $n = \deg(m_{\alpha,k}(x))$.

Proof: Recalling that $k(\alpha) \cong k[x] / \langle m_{\alpha,k}(x) \rangle$, we have

$$L \otimes_k k(\alpha) \cong L \otimes_k (k[x] / \langle m_{\alpha,k}(x) \rangle) \cong L[x] / \langle m_{\alpha,k}(x) \rangle .$$

In $L[x]$ there is a factorization $m_{\alpha,k}(x) = \prod_{i=1}^{m} (x - \alpha_i)^{n_i}$. By the Chinese Remainder Theorem, we get $L \otimes_k k(\alpha) \cong \overset{m}{\underset{i=1}{\times}} L[x] / \langle (x - \alpha_i)^{n_i} \rangle$, and so we get the first claim because if α is not separable then there are repeated roots. If α is separable then $n_i = 1$ for each i, following the final claim. □

Theorem 3.5. *Let E/k be an ast field extension, and A the intermediate field of the algebraic elements of E over k, then A/k is a separable finite (then simple) field extension. Moreover, if L realizes the ast property of E/k, i.e. $E \otimes_k L \cong \overset{n}{\underset{i=1}{\times}} L(x)$, then $[A : k] \leq n$. In particular, E is transcendental over k.*

Proof: Let be $\alpha \in A - k$ and $m_{\alpha,k}(x) \in k[x]$ its minimal polynomial.

By lemma 3.2 we can assume that L contains a splitting field of $m_{\alpha,k}(x)$.

Observe that 0 is the only nilpotent element of $\overset{n}{\underset{i=1}{\times}} L(x)$; it follows, by the lemma 3.4, that α is separable, and so A/k is separable.

The lemma 3.4 also provides a set of orthogonal idempotents associated to $m_{\alpha,k}(x)$, as many idempotents as its degree, then $\deg(m_{\alpha,k}(x)) \leq n$.

Now let be $\gamma \in A - k$ such that $m_{\gamma,k}(x)$ has maximal degree: $k(\gamma, \alpha)$ is a separable finite field extension of k then, by the Theorem of the Primitive Element, there exists $\delta \in A$ such that $k(\gamma, \alpha) = k(\delta)$. By maximality of the degree of $m_{\gamma,k}(x)$ we get $k(\gamma) = k(\delta)$, so $\alpha \in k(\gamma)$. It follows $A = k(\gamma)$.

Finally $E \neq A$: otherwise $E \otimes_k L$ would have finite dimension over L. □

Remark 3.6. Notice that theorem 3.5 can be proved without the hypothesis L/k algebraic.

Theorem 3.7. *Let be E/k be an ast field extension. If $t \in E$ is transcendental over k then $E/k(t)$ is a finite field extension. As a consequence, the transcendence degree of E/k is 1.*

Proof: Let us assume that L realizes the ast property of E/k, i.e. $E\otimes_k L\cong \bigtimes_{i=1}^{n} L\left(x\right)$. Let $t\in E$ be as in the statement and consider a $k\left(t\right)-$basis $\left\{b_j\right\}_{j\in J}$ of E.

Applying proposition 2.2 we have an injective composition of homomorphisms of $L-$algebras $\theta : L\left(t\right)\cong k\left(t\right)\otimes_k L \xrightarrow{\iota\otimes 1} E\otimes_k L$, where $\iota : k\left(t\right)\to E$ is the canonical inclusion, and homomorphisms of $L-$algebras

$$\vartheta_i : L\left(t\right) \xrightarrow{\theta} E\otimes_k L\cong \bigtimes_{i=1}^{n} L\left(x\right) \xrightarrow{\pi_i} L\left(x\right),$$

where π_i is the canonical projection corresponding to i.

Notice that $1\otimes 1\in k\left(t\right)\otimes_k L$ implies that $\vartheta_i\neq 0$ for each i, and so ϑ_i is injective for $i\in\left\{1,\ldots,n\right\}$.

It is easy to verify that $\left\{b_j\otimes 1\right\}_{j\in J}$ is a $L\left(t\right)-$basis of $E\otimes_k L$ for the structure of vector space induced by θ.

Notice that $t\notin L$ since L/k is algebraic. Then by theorem 3.3, for each i, $L\left(x\right)$ is finite dimensional over the image of ϑ_i, and so J has to be finite. □

Proof of corollary 1.1: By lemma 2.17 and theorem 3.2 of [5] we have that Z_G/k is an ast field extension, then applying theorems 3.5 and 3.7 we get that A_G/k is a finite field extension, Z_G/k is of transcendence degree one and, for $t\in Z_G$ transcendental over k, that $\left[Z_G : k\left(t\right)\right]<\infty$.

Since k is perfect there exists a separating transcendency basis for Z_G (see corollary of page 166 of [4]), i.e., we can choose t_0 transcendental over k such that $Z_G/k\left(t_0\right)$ is a separable field extension. By the Theorem of the Primitive Element $Z_G/k\left(t_0\right)$ is an algebraic simple extension.

Now let $\pi : E_G\to D_G$ be the canonical epimorphism and $T\in E_G$ such that $\pi\left(T\right)=t_0$. Then π induces an isomorphism of $k-$algebras between $k\left[T\right]$ and $k\left[t_0\right]$. By lemma 2.5.5 of [6] each element of $k\left[T\right]-\left\{0\right\}$ is invertible in E_G, and so there is a subring F of E_G that we can identify with $k\left(T\right)\cong k\left(t_0\right)$.

For the final claim we recall that generic modules are finitely generated over their endomorphisms rings and that D_G is finitely generated over Z_G (see [5]). □

Remark 3.8. With the hypothesis of corollary 1.1 it can be shown, applying Wedderburn's Principal Theorem, that $E_G=A\oplus B$ as $k-$vector spaces, A a subring of E_G, where $A\cong A_G$ as $k-$algebras.

Acknowledgements. We thank the support of grants PIFI-2013-31MSU0098J and P/PROFOCIE-2014-31MSU0098J-13

References

[1] R. Bautista, E. Pérez, L. Salmerón. *On semigenerically tame algebras over perfect fields.* Submitted.

[2] A. González, R. Jiménez, E. Pérez. *El Teorema de Lüroth.* Abstraction & Application **11** (2014) 52-56.

[3] E. Guerrero, E. Pérez. Álgebra Abstracta, de grupos a preliminares de la Teoría de Galois. Ediciones de la Universidad Autónoma de Yucatán, 2010.

[4] N. Jacobson. Lectures in Abstract Algebra, III. Theory of Fields and Galois Theory. Springer-Verlag, 1964. http://dx.doi.org/10.1007/978-1-4612-9872-4

[5] E. Pérez. *On semigeneric tameness and base field extension.* To appear in Glasgow Mathematical Journal.

[6] L. H. Rowen. Ring Theory (Student Edition). Academic Press, 1991. http://dx.doi.org/10.1016/b978-0-08-092548-6.50005-2

Received: January 11, 2015; Published: February 4, 2015