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INDEX OF MAXIMALITY AND GOSS ZETA FUNCTION

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ABSTRACT. In this article, we define the index of maximality $\mathfrak{m}(y)$ of a positive integer y, associated with the vanishing of certain power sums over $\mathbb{F}_q[T]$, related to the set $V_m(y)$ of "valid" decompositions of $y = X_1 + \cdots + X_m$ of length m. The index of maximality determines the maximum positive integer m for which the sets $V_m(y)$ are not empty. An algorithm is provided to find $V_i(y)$, $1 \leq i \leq \mathfrak{m}(y)$ explicitly.

The invariance, under some action, of the index of maximality $\mathfrak{m}(y)$ and of the property of divisibility by $q-1$ of $\ell_q(y)$, the sum of the q-adic digits of y, implies the invariance of the degree of Goss zeta function; it is illustrated here for two cases.

Finally, we generalize, to all q, the properties of an equivalence relation on $\mathbb{Z}_p$, which depends on the Newton polygon of the Goss zeta function.

1. Introduction. In [**7**], Sheats proved the Riemann hypothesis for the Goss zeta function on $\mathbb{F}_q[T]$; part of the proof depends on the assertion of Carlitz on the vanishing of certain power sums [**2**].

Let $\mathbb{N}$ be the set of non-negative integers, and put $\mathbb{Z}^+ = \mathbb{N}\backslash\{0\}$. Let p be a prime, and make $q = p^r$. Let A^+ be the set of monic polynomials in $A = \mathbb{F}_q[T]$. The power sums studied by Carlitz in [**2**] for positive integers y are:

$$S'_m(y) = \sum_{\substack{f \in A^+ \\ \deg f = m}} f^y.$$

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Definition 1.1. Let $U_m(y)$ be the set of m-tuples $(X_1, \ldots, X_m) \in \mathbb{N}^m$ whose terms sum to y and also satisfy the following conditions:

(1) there is no p-adic carry over in the sum $y = \sum X_j$;
(2) $X_j > 0$ and $(q-1) \mid X_j$ for $0 \leq j \leq m-1$.

Note that condition (2) does not set any restriction on X_m.

In **[2]**, Carlitz studied, among other things, the vanishing of the sums $S'_m(y)$ for positive integers y. He stated that $S'_m(y) \neq 0$ if and only if $U_{m+1}(y) \neq \emptyset$. During the proof of this statement, he wrote $S'_m(y)$ as follows:

$$S'_m(y) = \sum \binom{y}{X_0, \ldots, X_m} (-1)^m T^{X_1+2X_2+\cdots+mX_m},$$

where the sum is over all $(X_0, \ldots, X_m)$ such that $y = \sum X_j$ regardless of whether or not there is p-adic carry over. It involves this condition as follows: the Lucas theorem states that

$$\binom{y}{X_0, \ldots, X_m} = \frac{y!}{X_0! \cdots X_m!}$$

is not $0 (\operatorname{mod} p)$ if and only if there is no p-adic carry over in the sum $\sum X_j$. So we can write

$$S'_m(y) = \sum_{(X_0,\ldots,X_m)\in U_{m+1}(y)} \binom{y}{X_0, \ldots, X_m} (-1)^m T^{X_1+2X_2+\cdots+mX_m}.$$

Hence, if $U_{m+1}(y) = \emptyset$, then $S'_m(y) = 0$. For the converse, Carlitz stated without proof that:

> The degree $X_1 + 2X_2 + \cdots + mX_m$ of a monomial in the sum attains a single maximum when $(X_m, X_{m-1}, \ldots, X_0)$ is lexicographically larger than all the elements in $U_{m+1}(y)$.

Sheats in **[7]** shows a slightly different version of the above quotation.

Definition 1.2. For an m-tuple $X = (X_1, \ldots, X_m) \in \mathbb{N}^m$, the weight of X is defined as

$$wt(X) = X_1 + 2X_2 + \cdots + mX_m.$$

Given a finite subset $W \subset \mathbb{N}^m$, a tuple $O \in W$ is said to be optimal in W if $wt(O) \geq wt(X)$ for all $X \in W$. The greedy element of W is the tuple $(G_1, \ldots, G_m) \in W$ for which $(G_m, G_{m-1}, \ldots, G_1)$ is the largest, lexicographically. A composition of $y \in \mathbb{Z}^+$ is a tuple $X = (X_1, \ldots, X_m)$ of *positive* integers whose sum is y. We say that an m-tuple X is a valid composition if it also satisfies that there is no p-adic carry over in the sum $y = \sum_{i=1}^m X_i$ and $q-1 \mid X_j$ for $1 \leq j \leq m-1$. Define $V_m(y)$ as the set of all valid compositions of y of length m, and note that

$$V_m(y) = \{(X_1, \ldots, X_m) \in U_m(y) : X_m > 0\}.$$

Much of [**7**] is devoted to proving the following theorem.

Theorem 1.3. *If $V_m(y)$ is not empty, then it contains a single optimal element. Moreover, the optimal element is the greedy element of $V_m(y)$.*

Proof. See [**7,** page 125]. □

Theorem 1.3 implies its counterpart for $U_m(y)$, which is equivalent to the statement of Carlitz:

Theorem 1.4. *If $U_m(y)$ is not empty, then it contains a single optimal element. Moreover, the optimal element is the greedy element of $U_m(y)$.*

Proof. See [**7,** page 125]. □

It is a consequence of Theorem 1.4 that:

Theorem 1.5. *$S'_k(y) \neq 0$ if and only if $U_{k+1}(y) \neq \emptyset$.*

Proof. See [**7,** page 125]. □

This article introduces the index of maximality $\mathfrak{m}(y)$ for y a non-negative integer and shows its relationship with the sets $V_m(y)$. It provides an algorithm that can be implemented on a computer to find both $\mathfrak{m}(y)$ and $V_m(y)$.

Also, the invariance of the index under the action of the group $S_{(q)}$ and the action $i \mapsto p^i y$ are shown. As a simple application, using this index, we show how to calculate the degree of the Goss zeta function. The invariance of this index, under the actions described above, allows also to show the invariance of the degree of the Goss zeta function under the same actions.

In [**4**], Goss defines an equivalence relation on $\mathbb{Z}_p$ which depends on the zeta function and explicitly describes the equivalence classes in the case $q = p$. We generalize these results to all $q = p^r$.

Now, we will find conditions so that $V_m(y) \neq \emptyset$ in the case q prime.

2. Conditions for $V_m(y) \neq \emptyset$, case $q = p$. In [**3**] a simple and effective method is shown for calculating the greedy element $G = (G_1, \ldots, G_m)$ in $U_m(y)$: if $y = y_0 + y_1 p + \cdots + y_k p^k$ is the decomposition of y in base p,

$$G_1 = y_0 + y_1 p + \cdots + y_{j_1}^* p^{j_1}, y_{j_1}^* \leq y_{j_1},$$

where

$$y_0 + y_1 + \cdots + y_{j_1}^* = p - 1.$$

Notice that G_1 is the minimum value less than or equal to y such that this value is divisible by $p-1$. Then, G_2 is obtained (minimum) in the same way by considering the decomposition in base p of $y - G_1$. Then, G_3 is obtained (minimum) in the same way by considering the decomposition in base p of $y - (G_1 + G_2)$, and so on. Finally,

$$G_m = y - (G_1 + G_2 + \cdots + G_{m-1}).$$

This method of finding the greedy element is also useful to determine for which $m, V_m(y) \neq \emptyset$.

Let $\mathbb{Z}_p$ be the ring of p-adic integers and $y \in \mathbb{Z} \subseteq \mathbb{Z}_p$ a non-negative integer which can be written q-adically as $y = \sum_{i=0}^{w} y_i q^i$ where $0 \leq y_i \leq q-1$ for all i. Let $\ell_q(y) := \sum_i y_i$. If $y \in \mathbb{Z}_p$ is not a non-negative integer, we have $\ell_q(y) = \infty$.

Theorem 2.1. *Let $\mathfrak{m} = \mathfrak{m}(y) = \lceil \ell_p(y)/(p-1) \rceil$ where $\lceil k \rceil$ is the smallest integer greater than or equal to k. Then*

$$V_1(y), \ldots, V_{\mathfrak{m}}(y) \neq \emptyset \quad \textit{and} \quad V_{\mathfrak{m}+1}(y) = \emptyset.$$

Proof. Write $\ell_p(y) = \alpha(p-1) + r$, where $r = 0$ or $0 < r < p-1$. Suppose $r = 0$. Then $\mathfrak{m} = \ell_p(y)/(p-1)$. Note that, using the above algorithm, we can find $(X_1, \ldots, X_{\mathfrak{m}})$ with X_1 the minimum positive value divisible by $p-1$ and $\leq y$, X_2 the minimum positive value divisible by $p-1$ and $\leq y - X_1$, X_3 the minimum positive value divisible by $p-1$ and $\leq y - (X_1 + X_2)$, etc. Therefore, $V_{\mathfrak{m}}(y) \neq \emptyset$. However, since there are no extra digits, it is impossible to find an $X_{\mathfrak{m}+1} > 0$. Therefore, $V_{\mathfrak{m}+1}(y) = \emptyset$. Now, note that, if $0 < r < p-1$, then we can find $X_1, \ldots, X_{\mathfrak{m}-1}$ minimum and divisible by $p-1$, and greater than 0. And, since $r > 0$, we find $X_{\mathfrak{m}} > 0$ minimum and that is not divisible by $p-1$. Therefore, $V_{\mathfrak{m}}(y) \neq \emptyset$. Since it is impossible to find $X_{\mathfrak{m}+1} > 0$, we obtain that $V_{\mathfrak{m}+1}(y) = \emptyset$. This ends the proof. $\square$

Example 2.2. Let $q = 5$, $y = 1712 = 2\ 3\ 3\ 2\ 2_5$. Note that $\ell_q(y) = 12$. Then $\mathfrak{m} = \mathfrak{m}(y) = \lceil 12/4 \rceil = 3$. For example, $X = (2\ 2, 1\ 3\ 0\ 0, 2\ 2\ 0\ 0\ 0) = (12, 200, 1500)$. If $X = (X_1, X_2, X_3, X_4)$, then X_1, X_2, X_3 should have at least four digits 5-adic each. This would exhaust all the digits in y and, therefore, $X_4 = 0$. This is impossible.

In the next section, we find a generalization of Theorem 2.1 for the case of an arbitrary q.

3. Conditions for $V_m(y) \neq \emptyset$, case $q = p^r$. The index of maximality.

Definition 3.1. In set theory, a *multiset* (also called *bag*) is defined as a pair (A, m) where A is a set and $m : A \to \mathbb{N}$ is a function from A to $\mathbb{N}$. The set A is called the underlying set of elements. For each $a \in A$, the multiplicity of a is the number $m(a)$. It is common to write the function m as a set of ordered pairs $\{(a, m(a)) : a \in A\}$. For example, the multiset described as $\{a, b, b\}$ is written as $\{(a, 1), (b, 2)\}$. Informally, we describe a multiset with repeated elements explicitly and not as ordered pairs.

Now, we discuss a more efficient way of dealing with the conditions of Definition 1.1.

Definition 3.2. For $y \in \mathbb{N}$ define $\sigma(y)$ as the nondecreasing sequence of powers of p whose terms sum y.

Example 3.3. Let $p = 5$, $y = 11931$. By representing y in base 5, we obtain that $y = 3\ 4\ 2\ 1\ 1$. Then

$$\sigma(y) = \left\{1, 5, 5^2, 5^2, 5^4, 5^4, 5^4, 5^4, 5^5, 5^5, 5^5\right\}.$$

Hence,

> $(X_1, \ldots, X_m)$ satisfies condition (1) of Definition 1.1 if and only if $\{\sigma(X_1), \ldots, \sigma(X_m)\}$ is a partition (as a multiset) of $\sigma(y)$.

To deal with the p-adic digits when we add q-adic digits, we define the mapping $\Gamma : \mathbb{N} \to \mathbb{N}^r$ as follows. Given $y \in \mathbb{N}$ with p-adic expansion $y = \sum_{j\geq 0} y_j p^j$, define $\Gamma(y)$ as the column vector $(u_0, u_1, \ldots, u_{r-1})^t$ where u_i is the sum of all y_j such that $j \equiv i (\operatorname{mod} r)$. Let $\overline{\psi}_0 = (1, p, \ldots, p^{r-1})^t$ and $\langle \cdot, \cdot \rangle$ the usual inner product. Note that $\langle \overline{\psi}_0, \Gamma(y) \rangle$ is the sum of the q-adic digits of y.

A number y is divisible by $q - 1$ if and only if the sum of its q-adic digits is divisible by $q - 1$. Using this fact, it is clear that

> $(X_1, \ldots, X_m)$ satisfies condition (2) of Definition 1.1 if and only if $q - 1 \mid \langle \overline{\psi}_0, \Gamma(X_j) \rangle$ for $1 \leq j \leq m - 1$ and $X_1, X_2, \ldots, X_{m-1} > 0$.

Remark 3.4. If $(X_1, \ldots, X_m) \in V_m(y)$, then $\Gamma(X_1) + \cdots + \Gamma(X_m) = \Gamma(y)$ because in the sum there is no p-adic carry over.

Finally, we define a partial order: for two vectors $\overline{x} = (x_0, \ldots, x_{r-1})^t$ and $\overline{y} = (y_0, \ldots, y_{r-1})^t$ in $\mathbb{N}^r$, write that $\overline{x} \leq \overline{y}$ if and only if $x_i \leq y_i$ for $0 \leq i \leq r - 1$. It is important to note that $\overline{x} < \overline{y}$ means that $x_i \leq y_i$ for $0 \leq i \leq r - 1$ with $x_i < y_i$ for at least one i.

This leads to the following algorithm:

Algorithm 3.5 (Algorithm to calculate $V_m(y)$)**.** *To satisfy condition* (1) *of Definition* 1.1*, partition $\sigma(y)$ in m parts $(\Sigma_1, \ldots, \Sigma_m)$. Then, set X_i equal to the sum of the elements of Σ_i. To satisfy condition* (2)*, take X_i for $1 \leq i \leq m - 1$ such that $q - 1$ divides $\langle \overline{\psi}_0, \Gamma(X_i) \rangle$. Find the set of vectors $\overline{v} \in \mathbb{N}^r$ such that $\overline{v} < \Gamma(y)$ and $q - 1 \mid \langle \overline{\psi}_0, \overline{v} \rangle$.*

By having found this set of vectors, all the valid compositions are constructed.

Example 3.6. Let $p = 3$, $r = 2$, $m = 3, y = 4791$. Form $V_3(4791)$. Note that $4791 = 2\ 0\ 1\ 2\ 0\ 1\ 1\ 0_3$ and $4791 = 6\ 5\ 1\ 3_9$. Then $\Gamma(y) = (3,4)^t$. The only vector $\overline{v} = (v_0, v_1) \in \mathbb{N}^2$ such that $(v_0, v_1) < (3,4)^t$ and $8 \mid 1 \cdot v_0 + 3 \cdot v_1$ is $(2,2)^t$. Then $V_3(4791) = \emptyset$, because if the composition $(X_1, X_2, X_3) \in V_3(4791)$, then $\Gamma(X_1) = (2,2)$, $\Gamma(X_2) = (2,2)$ and $\Gamma(X_3) > 0$, which is impossible because of Remark *3.4*.

Example 3.7. Suppose that $m = 2$, and we have $V_2(4791) \neq \emptyset$. If (Σ_1, Σ_2) is the partition, since $\Gamma(X_1) + \Gamma(X_2) = \Gamma(y)$ and $\Gamma(X_1)$ must be divided by 8, the only option is $\Gamma(X_1) = (2,2)^t$. It is now clear that

$$X_1 \in \left\{ \begin{array}{l} 2\,0\,0\,1\,0\,1\,0\,0,\ 2\,0\,0\,2\,0\,0\,0\,0,\ 1\,0\,1\,1\,0\,1\,0\,0,\ 1\,0\,1\,2\,0\,0\,0\,0, \\ 1\,0\,0\,1\,0\,1\,1\,0,\ 1\,0\,0\,2\,0\,0\,1\,0,\ 1\,1\,0\,1\,1\,0,\ 1\,2\,0\,0\,1\,0 \end{array} \right\}$$

and, therefore,

$$V_2(4791) = \left\{ \begin{array}{l} (2\,0\,0\,1\,0\,1\,0\,0,\ 1\,1\,0\,0\,1\,0),\ (2\,0\,0\,2\,0\,0\,0\,0,\ 1\,0\,0\,1\,1\,0), \\ (1\,0\,1\,1\,0\,1\,0\,0,\ 1\,0\,0\,1\,0\,0\,1\,0), (1\,0\,1\,2\,0\,0\,0\,0,\ 1\,0\,0\,0\,0\,1\,1\,0), \\ (1\,0\,0\,1\,0\,1\,1\,0,\ 1\,0\,1\,1\,0\,0\,0\,0),\ (1\,0\,0\,2\,0\,0\,1\,0,\ 1\,0\,1\,0\,0\,1\,0\,0), \\ (1\,1\,0\,1\,1\,0,\ 2\,0\,0\,1\,0\,0\,0\,0),\ (1\,2\,0\,0\,1\,0,\ 2\,0\,0\,0\,0\,1\,0\,0\,) \end{array} \right\}.$$

Now, we formally define what a basis is.

Definition 3.8. A vector basis $\mathfrak{J}$ for y denoted by $\beta_{\mathfrak{J}} = \{\overline{v}_1, \ldots, \overline{v}_{s-1}\}$ is formed by vectors such that $\overline{v}_1, \ldots, \overline{v}_{s-1} \in \mathbb{N}^r \backslash \{0\}$ which satisfy the following conditions:

(1) $\overline{v}_1, \ldots, \overline{v}_{s-1} < \Gamma(y)$.
(2) $q - 1 \mid \langle \overline{\psi}_0, \overline{v}_i \rangle$ $1 \leq i \leq s-1$.

From Remark 3.4 and Definition 3.8, we have

Definition 3.9. Let $y \in \mathbb{Z}_p$ be a non-negative integer. We define the index of maximality of y, denoted by $\mathfrak{m}(y)$, as the maximum number of vectors $\overline{v}_i$ in the basis $\mathfrak{J}$ for y such that

$$\Gamma(y) - \left(\overline{v}_1 + \cdots + \overline{v}_{\mathfrak{m}(y)-1}\right) > 0.$$

If $y \in \mathbb{Z}_p - \{z \in \mathbb{Z}, z \geq 0\}$, put $\mathfrak{m}(y) = \infty$.

An immediate consequence of Definition 3.9 is the following theorem:

Theorem 3.10. *Let* $q = p^r$ *and* y *a non-negative integer. Let* $\mathfrak{m} = \mathfrak{m}(y)$ *be the index of maximality of* y. *Then*

$$V_1(y), \ldots, V_{\mathfrak{m}}(y) \neq \emptyset \quad \textit{and} \quad V_{\mathfrak{m}+1}(y) = \emptyset.$$

Proof. Let $\mathfrak{m} = \mathfrak{m}(y)$. Then choose $\overline{v}_1, \ldots, \overline{v}_{\mathfrak{m}-1} \in \beta_{\mathfrak{J}}$ such that $\Gamma(y) - (\overline{v}_1 + \cdots + \overline{v}_{\mathfrak{m}-1}) > 0$. Let X_i be such that $\Gamma(X_i) = \overline{v}_i$ for $1 \leq i \leq \mathfrak{m} - 1$ and $X_{\mathfrak{m}}$ such that $\Gamma(X_{\mathfrak{m}}) = \Gamma(y) - (\Gamma(X_1) + \cdots + \Gamma(X_{\mathfrak{m}-1}))$. As $q - 1 \mid \langle \overline{\psi}_0, \Gamma(X_i) \rangle$, then $q - 1 \mid X_i$ for $1 \leq i \leq \mathfrak{m} - 1$. From the fact that $\Gamma(X_{\mathfrak{m}}) > 0$, it is clear that $X_{\mathfrak{m}} \neq 0$. And, therefore, $(X_1, \ldots, X_{\mathfrak{m}}) \in V_{\mathfrak{m}}(y)$ and $V_{\mathfrak{m}}(y) \neq \emptyset$. Now suppose that $V_{\mathfrak{m}+1}(y) \neq \emptyset$. Let $(X_1, \ldots, X_{\mathfrak{m}+1}) \in V_{\mathfrak{m}+1}(y)$. Then $X_1, \ldots, X_{\mathfrak{m}}$ are divisible by $q-1$ and $X_{\mathfrak{m}+1} > 0$. That is, $\Gamma(X_1), \ldots, \Gamma(X_{\mathfrak{m}}) \in \beta_{\mathfrak{J}}$, $\Gamma(X_{\mathfrak{m}+1}) > 0$, and all of these Γ's are such that $\Gamma(y) - (\Gamma(X_1) + \cdots + \Gamma(X_{\mathfrak{m}})) > 0$ which contradicts the definition of $\mathfrak{m}$. □

Remark 3.11. Note that $\mathfrak{m}$, for which $V_{\mathfrak{m}}(y)$ is different from the empty set, agrees, in the case q prime, with the one defined in Theorem 2.1.

Example 3.12. (Rewriting Example 2.2). Let $q = 5$, $y = 1712 = 2\ 3\ 3\ 2\ 2_5$. Note that, as $r = 1$, we have that $\Gamma(y)$ is the sum of all p-adic digits of y, i.e., $\Gamma(y) = (12)$. Now, $\beta_{\mathfrak{J}} = \{4, 8\}$. It follows clearly that $\mathfrak{m}(y) = 3$. Therefore, $V_3(1712) \neq \emptyset$ and $V_4(1712) = \emptyset$.

Example 3.13. Let $p = 3$, $r = 2$, $q = 9$ and $y = 3281 = 1\ 1\ 1\ 1\ 1\ 1\ 1\ 2_3$. Note that $\ell_p(y) = 9$ and $\Gamma(y) = (5,4)^t$. Now, $\beta_{\mathfrak{J}} = \{(2,2)^t, (4,4)^t, (5,1)^t\}$. Hence,

$$\begin{aligned} (5,4)^t &= (2,2)^t + (2,2)^t + (1,0)^t \\ (5,4)^t &= (4,4)^t + (1,0)^t, \\ (5,4)^t &= (5,1)^t + (0,3)^t. \end{aligned} \tag{1}$$

Then $\mathfrak{m} = \mathfrak{m}(y) = 3$. It follows that $V_3(3281) \neq \emptyset$ and $V_4(3281) = \emptyset$. For example, $X = (1\ 0\ 1\ 2, 1\ 0\ 1\ 1\ 0\ 1\ 0\ 0,\ 1\ 0\ 0\ 0\ 0\ 0\ 0) = (32, 2520, 729) \in V_3(3281)$. Note that, because of (1) we have $V_4(3281) = \emptyset$.

Let $y \in \mathbb{Z}_p$. Write y q-adically as

$$\sum_{i=0}^{\infty} c_i q^i,$$

and define

$$\rho_*(y) := \sum_{i=0}^{\infty} c_i q^{\rho(i)},$$

where ρ is a permutation of $\mathbb{N}$. Let $S_{(q)}$ be the group of permutations of $\mathbb{Z}_p$ obtained as ρ varies over all permutations of $\mathbb{N}$. The action of this group over the p-adic integers was defined by Goss in **[5]** in his Eulerian search of a functional equation for the zeta function. There he shows, among others, the following basic properties of this action:

- (Semi-additivity). Let z, u, w be three p-adic integers with $z = u + w$ and where there is no carry over of q-adic digits. Then $\rho_*(z) = \rho_*(u) + \rho_*(w)$.
- Let y be an integer. Then $y \equiv \rho_*(y) \pmod{q-1}$.
- Let y be a non-negative integer. Then $\ell_q(y) = \ell_q(\rho_*(y))$.

After proving this, he shows by using the formula of Lucas, that the group $S_{(q)}$, $q = p^r$ has a natural relationship with binomial coefficients considered modulo p: Let X_1 and X_2 be two non-negative integers. Let $\rho_* \in S_{(q)}$. Then

$$\binom{X_1 + X_2}{X_1} \equiv \binom{\rho_*(X_1) + \rho_*(X_2)}{\rho_*(X_1)} \pmod{p}.$$

The result of this says that there is p-adic carry over in the sum of X_1 and X_2 if and only if it exists in the sum of $\rho_*(X_1)$ and $\rho_*(X_2)$.

The following theorem is an immediate consequence of the above, where we adapted the notation of Goss to our own notation.

Theorem 3.14 (**[5]**). *Let*

$$(X_1, \ldots, X_m) \in U_m(y) \quad (\textit{respectively } V_m(y)),$$

for $y \in \mathbb{Z}^+$. Then, for any $\rho_* \in S_{(q)}$,

$$(\rho_*(X_1), \ldots, \rho_*(X_m)) \in U_m(\rho_*(y)) \quad (\textit{respectively } V_m(\rho_*(y))).$$

This theorem implies the invariance of the index of maximality of a non-negative integer with respect to the action of the group $S_{(q)}$.

Corollary 3.15. *Let $q = p^r$ and y be a non-negative integer. Let $\mathfrak{m}(y)$ be the index of maximality of y. Then $\mathfrak{m}(y) = \mathfrak{m}(\rho_*(y))$, the index of maximality of $\rho_*(y)$, for any $\rho_* \in S_{(q)}$.*

The index of maximality does not change either if we change y by $p^i y$; this is shown in the following proposition.

Proposition 3.16. *Let y be a non-negative integer with index of maximality $\mathfrak{m}(y)$. Then $\mathfrak{m}(y) = \mathfrak{m}(p^i y)$ for all $i \in \mathbb{N}$.*

Proof. Let $e_0, \ldots, e_{n-1}$ denote the standard basis of column vectors for $\mathbb{R}^n$, define $R := [e_1, e_2, \ldots, e_{n-1}, e_0]$ to be the permutation matrix which rotates the coordinates of a vector to the right: $\mathrm{R}e_i = e_{i+1}$. Then, for any $i \in \mathbb{N}$, we have $R^i\Gamma(y) = \Gamma(p^i y)$. If

$$\beta_{\mathcal{J}}(y) = \{v_1, \ldots, v_{s-1}\},$$

from Definition 3.9 we obtain

$$\beta_{\mathcal{J}}\left(p^i y\right) = \left\{R^i v_1, \ldots, R^i v_{s-1}\right\}.$$

Therefore,

$$\mathfrak{m}(y) = \mathfrak{m}\left(p^i y\right). \qquad \square$$

4. The Goss zeta function. Let v_∞ be the valuation T^{-1}-adic over $K = \mathbb{F}_q(T)$. Then, the field of Laurent series $K_\infty := \mathbb{F}_q((T^{-1}))$ is the completion of K with respect to v. Let Ω be the completion of an algebraic closure of K_∞.

Definition 4.1. The Goss zeta function for $\mathbb{F}_q[T]$ is defined as

$$\zeta(z) = \sum_{f \in A^+} f^{-z},$$

where z is taken in $\Omega^* \times \mathbb{Z}_p$. Exponentiation is defined as follows: for a monic polynomial f, set $\langle f \rangle := fT^{-\deg f}$. Then, for $z = (x, y) \in \Omega^* \times \mathbb{Z}_p$, Goss defines

$$f^z = x^{\deg f} \langle f \rangle^y.$$

Goss showed that, by grouping terms of the same degree, the function $\zeta(z)$ is well defined for all $\Omega^* \times \mathbb{Z}_p$: for $z = (x, -y)$,

$$\zeta(z) = \zeta(x, -y) = \sum_{m \geq 0} x^{-m} \Bigg(\sum_{\substack{f \in A^+ \\ \deg f = m}} \langle f \rangle^y \Bigg) = \sum_{m \geq 0} a_m(y) \, x^{-m}.$$

From the definition of exponentiation, note that $f^{(T^m, m)} = f^m$ for any integer m.

Now, note that the sums studied by Carlitz for positive integers y,

$$S'_m(y) = \sum_{\substack{f \in A^+ \\ \deg f = m}} f^y$$

arise naturally as coefficients of the Goss zeta function. If $y \in \mathbb{N}$,

$$\begin{aligned} a_m(y) &= \sum_{\substack{f \in A^+ \\ \deg f = m}} \langle f \rangle^y = \sum_{\substack{f \in A^+ \\ \deg f = m}} \left(f T^{-\deg f} \right)^y = \sum_{\substack{f \in A^+ \\ \deg f = m}} f^y T^{-y \deg f} \\ &= \sum_{\substack{f \in A^+ \\ \deg f = m}} T^{-my} f^y = T^{-my} \sum_{\substack{f \in A^+ \\ \deg f = m}} f^y = T^{-my} S'_m(y). \end{aligned}$$

Then

$$v_\infty(a_m(y)) = my - \deg S'_m(y). \tag{2}$$

Now, if $y \in \mathbb{Z}_p \setminus \mathbb{N}$, we have that $y = y_0 + y_1 p + y_2 p^2 + \cdots + y_i p^i + \cdots$ has an infinite number of non-zero digits. Let $\widetilde{y}_t$ be the sum of the first $t+1$ terms in the p-adic expansion of y:

$$\widetilde{y}_t := \sum_{i=0}^{t} y_i p^i.$$

For any m, there exists t_m such that, if $t \geq t_m$,

$$v_\infty(a_m(y)) = v_\infty(a_m(\widetilde{y}_t)) = v_\infty(a_m(\widetilde{y}_{t_m})).$$

A proof of the assertion can be found in [**7,** Section 2]. Hence,

$$\begin{aligned} v_\infty(a_m(y)) &= m\widetilde{y}_{t_m} - \deg S'_m(\widetilde{y}_{t_m}) \\ &= m\widetilde{y}_t - \deg S'_m(\widetilde{y}_t) \; (\text{if } t \geq t_m). \end{aligned} \tag{3}$$

Thus, the case where $y \in \mathbb{Z}_p \setminus \mathbb{N}$ reduces to the case where it is a positive integer.

Our first result is related to the valuation at ∞ of the zeros of $\zeta(x,-y)$.

Proposition 4.2. *We have $v_\infty(a_m(y)) \equiv 0 (\operatorname{mod} q-1)$ for all m, y.*

Proof. Let $y \in \mathbb{N}$, and suppose that $S'_m(y) \neq 0$. From (2) and the proof of Theorem 1.1 of [**7,** page 123] we have $v_\infty(a_m(y)) = my - \deg S'_m(y)$ and, if $G = (G_1, \ldots, G_m)$ is the greedy element in $U_m(y)$, then

$$\deg S'_m(y) = G_1 + 2G_2 + \cdots + mG_m.$$

Therefore,

$$\begin{aligned} v_\infty(a_m(y)) &= my - \deg S'_m(y) = my - (G_1 + 2G_2 + \cdots + mG_m) \\ &\equiv m(y - G_m) = m(G_1 + \cdots + G_{m-1}) \\ &\equiv 0 (\operatorname{mod} q-1). \end{aligned}$$

The case $y \in \mathbb{Z}_p \setminus \mathbb{N}$ follows immediately from (3). □

Corollary 4.3. *Let $\alpha \in K$ a zero of $\zeta(x,-y)$ for $y \in \mathbb{Z}_p$ fixed. Then $v_\infty(\alpha)$ is positive and divisible by $q-1$.*

Proof. Suppose that $v_\infty(\alpha) \leq 0$. Then

$$v_\infty\left(\sum_{i=0}^{k} a_i(y)\,\alpha^{-i}\right) = v_\infty\left(1 + a_1(y)\,\alpha^{-1} + \cdots + a_k(y)\,\alpha^{-k}\right) = 0,$$

because $v_\infty(a_i(y)) - iv_\infty(\alpha) > 0$ for $i \geq 1$ since $v_\infty(a_i(y)) > 0$. On the other hand,

$$v_\infty\left(\lim_{k\to\infty}\sum_{i=0}^{k} a_i(y)\,\alpha^{-i}\right) = \lim_{k\to\infty} v_\infty\left(\sum_{i=0}^{k} a_i(y)\,\alpha^{-i}\right) = \infty$$

since α is a root. Hence, $v_\infty(\alpha) > 0$. Now, [**7**] shows that the Newton polygon of $\zeta(z)$ has only segments of vertical length 1. Hence, the valuations of the zeros are precisely the slopes

$$\lambda_y(m) = v_\infty(a_m(y)) - v_\infty(a_{m-1}(y)) = v_\infty(\alpha)$$

and, by Proposition 4.2,

$$\lambda_y(m) \equiv 0 \pmod{q-1}. \qquad \square$$

Goss in [**4**] shows that all zeros of $\zeta_{\mathbb{F}_p[T]}$ are near-trivial via the lemmas of Hensel and Krasner. As preliminary results, he finds the exact degree of the zeta function in the case q prime when y is a positive integer (this follows also from a theorem proved by Lee, a student of Carlitz, in [**6,** Theorem 3.2]), and properties of its coefficients and zeros equal those shown in Proposition 4.2 and Corollary 4.3.

Now, we show how to calculate the degree of $\zeta(x,-y)$, using the index of maximality.

Theorem 4.4.

(i) *Let $q = p^r$ and y be a non-negative integer. Let $\mathfrak{m} = \mathfrak{m}(y)$ be the index of maximality of y. Then the degree d in x^{-1} of $\zeta(x,-y)$ is*

$$d = \begin{cases} \mathfrak{m} - 1 & \text{if } q-1 \nmid \ell_q(y), \\ \mathfrak{m} & \text{if } q-1 \mid \ell_q(y). \end{cases}$$

(ii) *Let $y \in \mathbb{Z}_p$. Then the Newton polygon of $\zeta(x,-y)$ has at least n different slopes if and only if $d \geq n$.*

Proof. From Proposition 2.1 and Proposition 3.10, it follows that

$$V_{\mathfrak{m}}(y) \neq \emptyset \quad \text{and} \quad V_{\mathfrak{m}+1}(y) = \emptyset.$$

Note that, if $q-1 \nmid y$, $V_{\mathfrak{m}}(y) = U_{\mathfrak{m}}(y)$. Then $U_{\mathfrak{m}}(y) \neq \emptyset$ and $U_{\mathfrak{m}+1}(y) = \emptyset$. By Theorem 1.5, $S'_{\mathfrak{m}-1} \neq 0$ and $S'_{\mathfrak{m}} = 0$. If $q-1 \mid y$, $U_{\mathfrak{m}+1}(y)$ is essentially equal to $V_{\mathfrak{m}}(y)$ via the map $(X_1,\ldots,X_{\mathfrak{m}+1}) \mapsto (X_1,\ldots,X_{\mathfrak{m}} + X_{\mathfrak{m}+1})$. Then $V_{\mathfrak{m}}(y) \neq \emptyset$ implies that $U_{\mathfrak{m}+1}(y) \neq \emptyset$ and $V_{\mathfrak{m}+1}(y) = \emptyset$ implies that $U_{\mathfrak{m}+2}(y) = \emptyset$. Both equalities give as a result that $S'_{\mathfrak{m}} \neq 0$ and $S'_{\mathfrak{m}+1} = 0$. This ends the proof of the first part.

The second part is immediate from the first and Riemann hypothesis for the Goss zeta function. $\square$

This method of calculating the degree of the Goss zeta function and Corollary 3.15 gives another proof for the corollary that follows.

Corollary 4.5 ([**5**])**.** *Let y be a non-negative integer with associated zeta function $\zeta(x,-y)$. Let $\rho_* \in S_{(q)}$ arbitrary. Then $\zeta(x,-y)$ and $\zeta(x,-\rho_*(y))$ have the same degree in x^{-1}.*

Example 4.6 (Continuation of Example 3.12)**.** Let $q=5$, $y=1712=2\ 3\ 3\ 2\ 2_5$. Note that $\ell_p(1712)=12$ and $4\mid 12$. Then $d=3$, since $\mathfrak{m}(y)=3$. Moreover,

$$S_3'(y) = 2T^{3200} + 2T^{3180} + 2T^{3176} + T^{3160} + 4T^{3156} + \cdots ,$$
$$S_4'(y) = 0.$$

Example 4.7 (Continuation of Example 3.13)**.** Let $p=3$, $r=2$, $q=9$, $y=3281=4\ 4\ 4\ 5_9$. Note that $\ell_q(y)=17$ and $8\nmid 17$. Then $d=2$ because $\mathfrak{m}(y)=3$. Moreover,

$$S_2'(y) = T^{3978} + 2T^{3970} + 2T^{3898} + T^{3890} + T^{3762} + \cdots ,$$
$$S_3'(y) = 0.$$

The change of y for $p^i y$, $i\in\mathbb{N}$, does not change the degree of the zeta function. Since we are in characteristic p, this follows because $S_m(p^i y) = S_m(y)^{p^i}$. As an illustration, this can also be proved by using Theorem 4.4 as follows:

Proposition 4.8. *Let y a non-negative integer with associated zeta function $\zeta(x,-y)$. Then $\zeta(x,-y)$ and $\zeta(x,-p^i y)$ have the same degree in x^{-1} for all $i\in\mathbb{N}$.*

Proof. Clearly,

$$q-1 \mid l_q(y) \Longleftrightarrow q-1\mid y \Longleftrightarrow q-1 \mid p^i y \Longleftrightarrow q-1 \mid l_q\left(p^i y\right).$$

Then, by Proposition 3.16 and Theorem 4.4, we have that $\zeta(x,-y)$ and $\zeta(x,-p^i y)$ have the same degree in x^{-1} for all $i\in\mathbb{N}$. □

We can use the Newton polygon of $\zeta(x,y)$ to define an equivalence relation on $\mathbb{Z}_p$ in the following way: Let n be a fixed positive integer. Let $y_i\in\mathbb{Z}_p$, $i=1,2$, be such that the Newton polygons of $\zeta(x,y_i)$ has n finite slopes for each i. We say that $y_1\sim_n y_2$ if and only if the Newton polygons of $\zeta(x,y_1)$ and $\zeta(x,y_2)$ have the same first n segments. If

$y \in \mathbb{Z}_p$ does not have n finite slopes, then by definition, y is equivalent to itself.

It is clear that $\sim_n$ is an equivalence relationship which only depends on $\zeta(x, y)$ and n.

Now, let $y \in \mathbb{Z}_p$ be chosen so that $\mathfrak{m} = \mathfrak{m}(y) \geq n$ and we expand $-y$ p-adically as $\sum_{t=0}^{\infty} c_i p^i$ (where it may happen that all but finitely many of c_i vanish). Note that Theorem 4.4 tells us that $\mathfrak{m} = n$ or $\mathfrak{m} = n + 1$ depending upon the divisibility of $l_q(y)$.

Also, because of [**7,** Lemma 2.1], for all $m \geq 0$, t_m exists such that, if $t \geq t_m$, then

$$v_m(y) = v_m\left(\widetilde{y}_t\right),$$

where $v_m(y)$ is the valuation of the coefficient of x^{-m} in $\zeta(x, -y)$.

We find $t_{\mathfrak{m}}, t_{\mathfrak{m}-1}, \ldots, t_1$ (or $t_{\mathfrak{m}+1}, t_{\mathfrak{m}}, \ldots, t_1$), and let

$$t^*_{\mathfrak{m}} = \max\left\{t_{\mathfrak{m}}, t_{\mathfrak{m}-1}, \ldots, t_1\right\} \text{ or } t^*_{\mathfrak{m}} = \max\left\{t_{\mathfrak{m}+1}, t_{\mathfrak{m}}, \ldots, t_1\right\}.$$

Then, we define

$$y_n = \sum_{i=1}^{t^*_{\mathfrak{m}}} c_i p^i.$$

Proposition 4.9.

(1) *We have* $-y_n \sim_n y$.
(2) y_n *is the smallest element in the set of positive integers* i *with* $-i \sim_n y$.
(3) *Let* y *and* z *in* $\mathbb{Z}_p$. *Then* $y \sim_n z$ *if and only if* $y_n = z_n$.

Proof.

(1) We have

$$v_\infty\left(a_{\mathfrak{m}}\left(y\right)\right) = \mathfrak{m}G'_1 + \left(\mathfrak{m} - 1\right)G'_2 + \cdots + G'_{\mathfrak{m}}$$

and

$$v_\infty\left(a_{\mathfrak{m}}\left(y_n\right)\right) = \mathfrak{m}G_1 + \left(\mathfrak{m} - 1\right)G_2 + \cdots + G_{\mathfrak{m}}$$

with $(G'_1, \ldots, G'_{\mathfrak{m}}, G'_{\mathfrak{m}+1})$ the greedy element in $U_{\mathfrak{m}+1}(\widetilde{y}_t)$ and

$$(G_1, \ldots, G_{\mathfrak{m}}, G_{\mathfrak{m}+1})$$

the greedy element in $U_{\mathfrak{m}+1}(y_n)$. We will have finished if we show that both valuations are the same. We have, by Theorem 3.10, that $V_{\mathfrak{m}+1}(y) = V_{\mathfrak{m}+1}(y_n) = \emptyset$, so $G'_{\mathfrak{m}+1} = G_{\mathfrak{m}+1} = 0$. By Proposition 4.1 of [**7**], we have that $(G'_1, \ldots, G'_{\mathfrak{m}})$ is the greedy element in $V_{\mathfrak{m}}(\widetilde{y}_t - G'_{\mathfrak{m}+1})$ and $(G_1, \ldots, G_{\mathfrak{m}})$ is the greedy element in $V_{\mathfrak{m}}(y_n - G_{\mathfrak{m}+1})$. The equality follows from the definition of the index of maximality.

(2) Let i be a positive integer such that $-i \sim_n y$ and $i < y_n$. Because $\sim_n$ is an equivalence relation $-i \sim_n -y_n$, the Newton polygon of $\zeta(x, i)$ and $\zeta(x, y_n)$ have the same first n segments. Therefore, because of the preceding paragraph, the greedy element in $U_{\mathfrak{m}-1}(y_n - G_{\mathfrak{m}})$ is equal to the greedy element in $U_{\mathfrak{m}-1}(i - G'_{\mathfrak{m}})$. But this is a contradiction.

(3) It follows from the fact that $\sim_n$ is an equivalence relation and (2). $\square$

Thus, the equivalence classes under $\sim_n$ consisting of more than one element are in a one-to-one correspondence with the negative integers $-j$ described in the preceding proposition. Note that j is divisible by $q-1$.

For q arbitrary, Lee ([**6,** Lemma 7.1]), proved that if $m > l_q(y)/(q-1)$, then $S'_m(y) = 0$, so that the degree of $\zeta(x, -y)$ is less than or equal to $\lfloor l_q(y)/(q-1) \rfloor$, where $\lfloor\ \rfloor$ is the greatest integer function. Since the degree of $\zeta(x, -y)$ does not change, if we change y to p^i, $i \in \mathbb{N}$, the degree is less than or equal to $\lfloor l_q(p^i y)/(q-1) \rfloor$ for all $i \in \mathbb{N}$. But the set

$$\left\{ \left\lfloor l_q(p^i y)/(q-1) \right\rfloor : i \in \mathbb{N} \right\} = \left\{ \left\lfloor l_q(p^i y)/(q-1) \right\rfloor : 0 \le i \le n-1 \right\}.$$

Therefore, the degree of $\zeta(x, -y)$ is less than or equal to

$$\min_{0 \le i \le n-1} \left\lfloor l_q(p^i y)/(q-1) \right\rfloor.$$

Böeckle realized that, in fact, it follows from [**7**], although Sheats does not have this established in a specific way, that the above inequality is actually an equality. Therefore, the degree of Goss zeta function is given by the following simple formula:

$$\left\{ \left\lfloor \frac{\min_{0 \le i \le n-1} \ell_q\left(p^i y\right)}{q-1} \right\rfloor \right\}.$$

The second author learned this from Böckle at a seminar in March, 2009, while visiting the University of Arizona, Tucson. See [**1,** Prop. 10.21].

Note that, because the degree is invariant under the action of $S_{(q)}$, the formula of Böckle is also invariant under this action, i.e.,

$$\left\{ \left\lfloor \frac{\min_{0\le i\le n-1} \ell_q \left(p^i y\right)}{q-1} \right\rfloor \right\} = \left\{ \left\lfloor \frac{\min_{0\le i\le n-1} \ell_q \left(\rho_*(p^i y)\right)}{q-1} \right\rfloor \right\}.$$

Remark 4.10. The action given in Proposition 4.8, $y \mapsto p^i y, i \in \mathbb{N}$, under which the degree is invariant, is not necessarily one of the actions in $S_{(q)}$, since $\ell_q(p^i y)$ could be different from $\ell_q(y)$. The invariance of the index of maximality and of the property of divisibility by $q-1$ of $\ell_q(y)$ under some action implies the invariance of the degree of the Goss zeta function under this action.

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